Graph Theory: Problem set 8

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1. Let $G = (V, E)$ denote a graph with the independence number $\alpha(G)$ at most k. Show that G contains a cycle C of length at least $c_k|V|$ for some constant c_k depending only on k. (*hint: find an induced subgraph of G with the minimal degree high enough*)

2. Let us define the Cartesian product of the two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as the graph $G = G_1 \times G_2 = (V_1 \times V_2, \{(u_1, u_2)(v_1, v_2)|\ u_1v_1 \in E_1 \text{ or } u_2v_2 \in E_2\})$. Prove that $\chi(G_1)\chi(G_2) \geq \chi(G)$.

3. Consider a set C of non touching and non self-crossing n curves in the plane such that each curve is properly crossed by another curve at least twice, and no three curves have a common intersection. We define an undirected plane graph G whose vertex set is the set of intersections between curves and two vertices are joined by an edge if they are two consecutive crossing on a curve.

By orienting a curve containing vertices $v_1, \dots v_k$ we mean assigning an orientation to the edges v_iv_{i+1}, $1 \leq i \leq k-1$, such that the in-degree and out-degree of any vertex $v_2, \dots v_{k-1}$ is 1, if we restrict ourselves to the path $v_1 \dots v_k$.

Prove that if we can orient the curves in C in a way such that the resulting oriented graph does not contain a directed cycle, then there must be at least two vertices in G of degree 2.

4. $*$ Consider a drawing of a planar graph all of whose faces, including the outer one, are triangular (i.e. have 3 vertices). To each vertex we assign, quite arbitrarily, one of the labels 1,2 and 3. Prove that there are an even number of faces whose vertices get all 3 labels.

5. Let G be a graph $G = (V, E)$. We define the line graph $G' = (E, E')$ of G to be the graph whose vertex set is simply the edge set of G and two vertices in G' are joined by an edge if their corresponding edges in G share a vertex. More formally, $ef \in E'$ if there exists $u, v, w \in V$ such that $e = uv$ and $f = uw$.

Show that $\chi(H)$ is $\omega(H)$ or $\omega(H) + 1$ for any line graph H.