

# Graph Theory: Problem set 7

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1. An  $n$  by  $n$  matrix with entries from  $\{1, 2 \dots n\}$  is called a *Latin square*, if every element of  $\{1, 2 \dots n\}$  appears exactly once in each column, and exactly once in each row. Recast the problem of constructing Latin squares as a coloring problem.

We define a graph  $G = (\{1, 2 \dots n\} \times \{1, 2 \dots n\}, E)$ , where  $uv \in E$  iff  $u = (a, b)$  and  $v = (a, c)$ , or  $u = (b, a)$  and  $v = (c, a)$ . The problem of constructing Latin square is now equivalent to the problem of coloring  $G$  by  $n$  colors.

2. By an *outerplanar graph* we understand a planar graph that can be embedded in the plane so that it has all the vertices on the outer face.

Show that the chromatic number of an outerplanar graph is at most three.

Let  $G$  be an outerplanar graph embedded in the plane so that it has all the vertices on the outer face. First suppose that  $G$  is two-connected. If  $G$  is a cycle we are done, since any cycle can be properly colored by three colors. Otherwise  $G$  contains an edge  $uv$  such that  $u$  and  $v$  do not follow each other on the outerface of  $G$ . Moreover,  $u$  and  $v$  is a cut-set of  $G$ , which divides  $G$  into two components  $C_1$  and  $C_2$ . Let  $G_i$ , for  $i = 1, 2$ , denote the induced subgraphs of  $G$  on the vertex set  $V(C_i) \cup \{u, v\}$ . By induction we can color  $G_1$  and  $G_2$  with three colors so that in both colorings  $u$  and  $v$  get the color 1 and 2, let's say. Putting these coloring together in order to color  $G$  finishes the proof.

If  $G$  is not connected or contains a cut-vertex one can use the next exercise to finish the proof.

3. Let  $G$  denote a graph that contains a cycle. Show that  $\chi(G) = \max_{C \in \mathcal{C}(G)} \chi(C)$ , where  $\mathcal{C}(G)$  is the set consisting of the vertex two-connected components (subgraphs) in  $G$ .

W.l.o.g we can assume that  $G$  is connected. If  $G$  is two-connected we are done. Otherwise  $G$  contains a cut-vertex  $v$ . Removal of  $v$  divides  $G$  into  $c$  connected components  $C_1, \dots, C_c$ . Let  $G_i$ , for  $i = 1, \dots, c$ , denote the induced subgraphs of  $G$  on the vertex set  $V(C_i) \cup \{v\}$ . By induction we can color each  $G_i$  with  $\max_{C \in \mathcal{C}(G)} \chi(C)$  colors so that in each coloring  $v$  gets the color 1, let's say. Putting these coloring together in order to color  $G$  finishes the proof.

4. Let  $k$  denote a natural number. Describe a construction of a triangle-free graph with chromatic number  $k$ .

*Hint:* Let  $G = (V, E)$  denote a graph. Let  $V_0 = \{u' \mid u \in V\}$ , so that  $V_0 \cap V = \emptyset$  (think of  $V_0$  as of a copy of  $V$ ). Using  $G$  we construct the graph  $G' = (V', E')$  as follows:  $V' = V \cup V_0 \cup \{z\}$ ,  $z \notin V \cup V_0$ ,  $E' = E \cup \{u'v \mid uv \in E\} \cup \{zu' \mid u' \in V_0\}$ . Show that  $\chi(G') = \chi(G) + 1$ .

It is easy to see that  $\chi(G) \leq \chi(G') \leq \chi(G) + 1$ . Indeed, we can take a coloring  $c$  of  $G$  witnessing  $\chi(G)$  and color each pair  $u \in V$  and  $u' \in V_0$  with the color  $c(v)$ . Then we color  $z$  with one additional color.

Thus, it remains to show that  $\chi(G') \neq \chi(G)$ . This reduces to showing that in a proper coloring of  $G'$  we have to use  $\chi(G)$  colors for coloring  $V_0$ . For the sake of contradiction we assume that in a proper coloring of  $G'$  we use less than  $\chi(G)$  colors (out of those that are used to color  $V$ ) to color  $V_0$ . Let  $f$  denote the color that is missing in  $c(V_0)$  i.e.  $f \in c(V) \setminus c(V_0)$ . Let  $V^f \subseteq V$  denote the color class corresponding to the color  $f$ . Let  $V_0^f \subseteq V_0$  denote the subset of  $V_0$  so that  $u \in V^f$  iff  $u' \in V_0^f$ . Observe that we can recolor each vertex  $u$  in  $V^f$  with the color of  $u'$ . Indeed, the neighbours of  $u$  form a subset of the set of neighbours of  $u'$ , and  $u$  and  $u'$  are not joined by an edge. Hence, we can properly color  $G$  with  $\chi(G) - 1$  colors (contradiction).

Argue that  $G'$  is triangle free if  $G$  is triangle free.

5. We define the line graph  $G' = (E, E')$  of  $G$  to be the graph whose vertex set is simply the edge set of  $G$  and two vertices in  $G'$  are joined by an edge if their corresponding edges in  $G$  share a vertex. More formally,  $ef \in E'$  if there exists  $u, v, w \in V$  such that  $e = uv$  and  $f = uw$ .

Prove that the line graph of  $K^n$  has chromatic number either  $n - 1$  or  $n$ .

Prove that for odd  $n$  the answer is  $n$ , and for even  $n$  the answer is  $n - 1$ .

Let us draw the vertices of  $K^n$  as the vertices of the regular convex  $n$ -gon, and represent the edges of  $K^n$  by straight line segments.

Since all the edges incident to a single vertex must receive different colors, clearly,  $n - 1$  colors is necessary

On the other hand we can color parallel edges (edges whose corresponding line segments belong to parallel lines) by the same color. It is easy to see that there are at most  $n$  directions, which a diagonal segment of a regular convex  $n$ -gon can have.

Indeed, if  $n$  is even, the diagonal segment divides the set of remaining vertices into two parts having the same parity. If both parts are even our diagonal segment has the same direction as certain two opposite diagonals joining two vertices at distance two in the cyclic order along the defining circle of the  $n$ -gon. If both parts are odd our diagonal segment has the same direction as certain two opposite sides of our  $n$ -gon. As we have

altogether  $2n$  sides and diagonals of length two, having  $n$  distinct direction, in this case we are done.

The case when  $n$  is odd can be treated analogously.

Hence,  $n$  colors is enough.

6. \* Prove that you can color the integer lattice  $\mathbb{Z}^2$  with 4 colors, such that for any two points  $u, v \in \mathbb{Z}^2$  that can *see each other* (i.e. the interior of the segment  $\overline{uv}$  does not contain a point from  $\mathbb{Z}^2$ ) have distinct colors.

Hint: for each lattice point  $p \in \mathbb{Z}^2$ , let  $p = (x, y)$ . We color  $p$  as follows.

- (a) Color  $p$  Red if  $x$  is even and  $y$  is even,
- (b) color  $p$  Blue if  $x$  is even and  $y$  is odd,
- (c) color  $p$  Green if  $x$  is odd and  $y$  is even,
- (d) color  $p$  Yellow if  $x$  is odd and  $y$  is odd.

Argue that the midpoint of any two points in the same color class must be a lattice point.