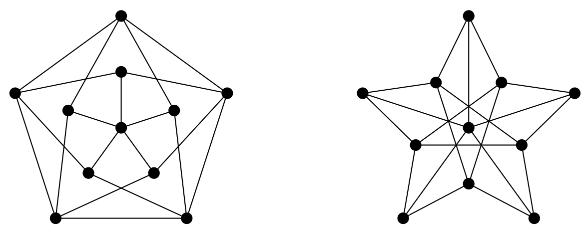
Graph theory - problem set 7

April 6, 2017

Exercises

1. Determine the girth and circumference of the following graphs. In particular, do they have Hamilton cycles?



2. Determine the length of the longest path and the longest cycle in the complete bipartite graph $K_{m,n}$, for all m, n.

3. In this exercise we show that the sufficient conditions for Hamiltonicity that we saw in the lecture are "tight" in some sense.

   (a) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has $\binom{n-1}{2} + 1$ edges.

   (b) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has minimum degree $\lceil \frac{n}{2} \rceil - 1$.

4. (a) Show that a k-regular graph with girth 5 must have at least $k^2 + 1$ vertices.

   (b) Find a k-regular graph with girth 5 and $k^2 + 1$ vertices for $k = 2, 3$.

   (c) Show that a k-regular graph with girth 4 must have at least $2k$ vertices.

Problems

5. Show that the proof of Dirac's Theorem also proves the following statement (called Ore's theorem): If for all non-adjacent $u, v \in V(G)$ we have $d(u) + d(v) \geq |V(G)|$, then G has a Hamilton cycle.

6. Use the previous problem to give a short proof of the fact that any graph G with $|E(G)| > \binom{|V(G)|-1}{2} + 1$ has a Hamilton cycle.

7. Let G be a connected graph on n vertices with minimum degree δ. Show that

   (a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ, and

   (b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamiltonian path.

8. An oriented complete graph (i.e. one that has exactly one directed edge between any pair of vertices) is called a *tournament*. Show that every tournament contains a (directed) Hamilton path.