

Graph Theory: Problem set 6 (hints)

March 28, 2011

1. Show that the complement of a simple planar graph on at least eleven vertices is nonplanar.

From $m_1 \leq 3n - 6$, $m_2 \leq 3n - 6$ and $m_1 + m_2 = \binom{n}{2}$, it follows that $n \leq 10$.

2. (a) Let S be a set of n points in the plane, where $n \geq 3$ and the distance between any two points of S is at least one. Show that no more than $3n - 6$ pairs of points of S can be at distance exactly one.
(b) Show by a construction that the leading term in the above bound cannot be improved.

For (a) note that the corresponding graph is planar. For (b) use the triangular grid.

3. Let \mathcal{L} be a finite set of lines in the plane, no two of which are parallel and not all of which are concurrent. Using Euler's Formula, show that some point is the point of intersection of precisely two lines of \mathcal{L} .

Imitate the proofs from the Euclidean case to derive Euler's formula for the projective plane: $n - m + \ell = 1$. As a consequence, prove the inequality $m \leq 3n - 3$. Alternatively, introduce a suitable planar graph (add a large circle!) and refine the inequality $m \leq 3n - 6$.

4. Show that the (3×3) -grid has a K_4 -minor.

Let $V = \{(a, b) : 1 \leq a, b \leq 3\}$. Consider the partition $V_1 = \{11, 12\}$, $V_2 = \{21, 31, 32, 33\}$, $V_3 = \{22\}$, $V_4 = \{13, 23\}$.

5. Prove that every simple 3-connected planar graph admits a planar embedding all of whose faces are bounded by *convex* polygons (in particular, all edges are straight-line segments).

Use the same recursive algorithm indicated in the proof of Kuratowski's theorem.

6. *¹ Show that a graph is outerplanar if and only if it has neither a K_4 -minor nor a $K_{2,3}$ -minor. (Recall that, by definition, a graph is *outerplanar* if it admits an embedding in the plane such that all vertices lie on the boundary of the outer/unbounded face.)

Imitate the proof of Kuratowski's theorem.

¹extra credit