Graph Theory: Problem set 5

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1. *Solution:* The edges incident to a vertex v of minimum degree form an edge cut, hence $\kappa'(G) \leq \delta(G)$. It remains to show that $\kappa(G) \leq \kappa'(G)$.

   Clearly $\kappa(G) \leq n-1$. Consider the smallest edge cut $[S, \overline{S}]$. If every vertex of S is adjacent to every vertex of $\overline{S}$, then $|[S, \overline{S}]| = |S||\overline{S}| \geq n-1 \geq \kappa(G)$, and we are done.

   Otherwise, we choose $x \in S$ and $y \in \overline{S}$ such that x is not adjacent to y. Let T consist of all neighbors of x in $\overline{S}$ and all vertices of $S - \{x\}$ with neighbors in $\overline{S}$. Every x, y path passes through T, so T is a separating set. Also picking the edges from x to $T \cap \overline{S}$ and one edge from each vertex of $T \cap S$ to $\overline{S}$ yield $|T|$ distinct edges of $[S, \overline{S}]$. Hence $\kappa'(G) = |[S, \overline{S}]| \geq |T| \geq \kappa(G)$.

2. *Solution:* Let S be a minimum vertex cut ($|S| = \kappa(G)$). Since $\kappa(G) \leq \kappa'(G)$ always, we need only provide an edge cut of size $|S|$. Let H_1, H_2 be two components of $G - S$. Since S is a minimum vertex cut, each $v \in S$ has a neighbor in H_1 and a neighbor in H_2. Since G is 3-regular, v cannot have two neighbors in H_1 and two in H_2. For each $v \in S$, delete the edge from v to a member of H_1, H_2 where v has only one neighbor.

   These $\kappa(G)$ edges break all paths from H_1 to H_2 except in the case where a path can enter S via v_1 and leave via v_2 (vertices in S). In this case, we delete the edge to H_1 for both v_1 and v_2 to break all paths from H_1 to H_2 through $\{v_1, v_2\}$.

3. Show that a planar graph on n vertices none of which faces is triangular can have at most $2n - 4$ edges.

   Let $G = (V, E)$ denote a plane graph (a good embedding of a planar graph) such that G does not contain a triangle. Let F denote the set of faces of G. Let us count the number of pairs (e, f), $e \in E$ and $f \in F$, such that e lies on f. Since each face is of size at least 4, and each edge lies on exactly two faces, we have the following inequality: $4|F| \geq 2|E|$. Hence by Euler's polyhedral formula $4|F| = 4|E| - 4|V| + 8 \geq 2|E|$. By reordering the terms we get the inequality.

4. Let us denote by v_i the number of degree i vertices in a planar graph G on at least 3 vertices. Prove the following inequality $12 \leq \sum_{i=1}^{\infty}(6-i)v_i$.

   Observe that the inequality is equivalent to the following inequality $12 \leq 6|V| - 2|E|$, which is true by Euler's polyhedral formula.

5. *Solution:* (a) implies (b). A face boundary consists of clsed walks. Every odd closed walk contains an odd cycle. Therefore, in a bipartite plane graph the contributions to the length of faces are all even.

(b) implies (a). Let C be a cycle in G. Since G has no crossings, C is laid out as a simple closed curve; let F be the region enclosed by C. Every region of G is wholly within F or wholly outside F. If we sum the face lengths for the regions inside F, we obtain an even number, since each face length is even. This sum counts each edge of C once. It also counts each edge inside F twice, since each such edge belongs twice to faces in F. Hence the parity of the length of C is the same as the parity of the full sum, which is even.

(b) iff (c). The dual graph is connected, and its vertex degrees are the face lengths of G.

6. $*$ Let G be a plane graph on n vertices, which has no face shorter than 4 and no two faces of length at most 5 that share an edge. Prove that G can have at most $\frac{12}{7}n$ edges.

Let n, m, f denote the number of vertices, edges and faces in G, respectively. Let f_5 denote the number of faces of G of length at most 5. By double counting the edges we have: $2m \geq 4f_5 + 6(f - f_5)$. Clearly, f_5 cannot exceed $m/4$. Indeed, if $f_5 > m/4$, then $4f_5 > m$. Thus, by pigeon-hole principle at least two faces of size at most 5 share an edge (contradiction). By above we get:

$$\begin{aligned} 2m &\geq 4f_5 + 6(f - f_5) \\ &= 6f - 2f_5 \\ &\geq 6f - \frac{m}{2} \end{aligned}$$

By rearranging the terms we get $5m \geq 12f$. Finally, the claim follows by Euler formula:

$$\begin{aligned} 12n + 5m &\geq 12n + 12f \\ &= 12m + 24 \end{aligned}$$

7. We call the planar graph *outer-planar* if it can be embedded into $\mathbb{R}^2$ in a way that all of its vertices lie on the outer face.

Show that an outer-planar graph G on $n > 2$ vertices can contain at most $2n - 3$ edges.

We proceed by induction on $|G|$. If $n = 3$ we are done. For inductive step observe that we can suppose that all vertices of G have degree at least 3, because if it is not the case by removing a vertex v of degree one or two we can apply inductive hypothesis as follows $|E(G)| \leq |E(G \setminus v)| + 2 \leq 2(n-1) - 3 + 2 = 2(n) - 3$.

We show that the above assumption (all vertices of G have degree at least 3) leads to a contradiction.

Let $S = v_1, \ldots v_m$ denote the sequence of vertices of G in the order as they appear on the outer-face (a vertex may appear more than once in this sequence), i.e. $v_i v_{i+1}$ is an edge.

Let $S' = (v = v_i), v_{i+1} \dots v_j = v$ be a subsequence of S not containing three occurrences of v, where no element except v appears twice. As we have rule out the possibility having a vertex of degree 1 the length of S' is at least four. Thus, S' corresponds to an outer face of length at least 3. Consider an edge $e = v_k v_l \in E(G)$, $i < k+1 < l \leq j$, minimizing $l - k$. Such an edge must exists by the minimum degree 3 in G. Think of e as of an diagonal with the minimal length. Consider the vertex v_{k+1}. This vertex has degree at least three. However, for the edge $v_{k+1}v_{l'}$, $l' \neq k+2, k$, we have $l' < k$ or $l' > l$ by the minimality of $l - k$ (see Figure 1 for an illustration), forcing us to make an edge crossing (contradiction).

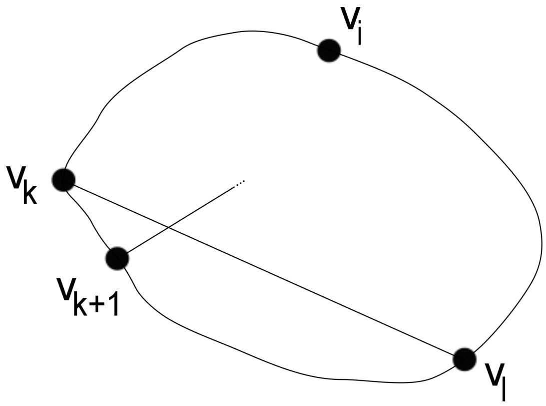

Figure 1: problem 5.6