Graph Theory: Problem set 5

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1. Let G be a graph $G = (V, E)$. We define the line graph $G' = (E, E')$ of G to be the graph whose vertex set is simply the edge set of G and two vertices in G' are joined by an edge if their corresponding edges in G share a vertex. More formally, $ef \in E'$ if there exists $u, v, w \in V$ such that $e = uv$ and $f = uw$.

Prove that if G is l-edge connected then G' is l (vertex) connected.

2. Show that every planar graph with n vertices which has no triangular face has at most $2n - 4$ edges.

3. Let us denote by v_i the number of degree i vertices in a planar graph G on at least 3 vertices. Prove the following inequality $12 \leq \sum_{i=1}^{\infty}(6 - i)v_i$.

4. Let G be a connected planar graph whose all internal vertices (not lying on the outerface) have an even degree and whose every pair of faces share at most one edge.

Prove that one can color its faces by two colors such that any two faces (except the outermost face) sharing an edge receive different colors. Here is the proposed line of thoughts.

a) Construct a graph G' each of which vertex corresponds to an internal face of G and whose pair of vertices is joined by an edge iff its respective faces share an edge.

b) Prove that G' is bipartite using the fact that each cycle in a 2-connected planar graph can be obtained as a symmetric difference of the face cycles, i.e $\bigtriangledown_{\alpha \in A} E_{f_\alpha}$, where E_{f_α} is the set of edges belonging to the face f_α of G'.

5. $*$ Let G be a plane graph on n vertices, which has no face shorter than 4 and no two faces of length at most 5 that share an edge. Prove that G can have at most $\frac{12}{7}n$ edges.

6. Show that any graph can be embedded into $\mathbb{R}^3$ without edge crossing such that we represent the vertices by points and edges by straight line segments between vertices. Can you explicitly define a desired set of n points in $\mathbb{R}^3$, which can represent the vertices of any graph on n vertices ?

7. We call the planar graph *outerplanar* if it can be embedded into $\mathbb{R}^2$ in a way that all of its vertices lie on the outer face.

Show that an outerplanar graph on $n > 2$ vertices can contain at most $2n - 3$ edges.