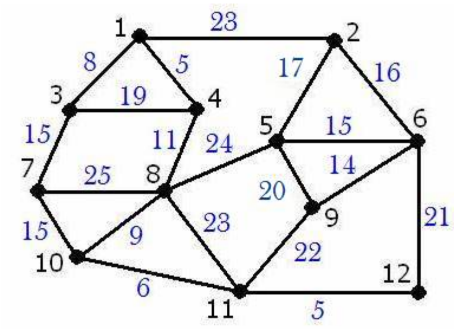
Problem Set 5.

5-1: Let G be a connected weighted graph. Assume it has at least one cycle, and let edge e be an edge that has strictly greater cost than all other edges in that cycle. Show that e does not belong to any minimal weight spanning tree of G.

5-2: Prove that any connected graph with distinct weights assigned to the edges has a unique minimal weight spanning tree.

5-3: Show that Kruskal's algorithm produces a minimal weight spanning tree even if some of the weights on the edges may be negative.

5-4: Apply Kruskal's algorithm to the following graph to obtain a minimum spanning tree:



5-5: The following is called Prim's algorithm:

- *Initialize a tree with a single vertex, chosen arbitrarily from the graph.*
- *Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and add it to the tree.*
- *Repeat step 2 (until all vertices are in the tree).*

Prove that, given a graph G as input, the output of Prim's algorithm is a minimum spanning tree of G. Apply Prim's algorithm to the graph from exercise 4.

5-6: Let $T_1, \ldots, T_k$ be trees on disjoint sets of points and $V = V(T_1) \cup \ldots \cup V(T_k)$. What is the number of labeled trees on V containing $T_1, \ldots, T_k$?

5-7: What is the number $F(n, k)$ of forests on the vertex set $[n]$ having k components and such that $1, \ldots, k$ belong to distinct components?

5-8: * There are 5 fox holes aligned in a line, and an *invisible fox* is hiding in one of them. Your goal is to shoot it by shooting in the holes. After each shot, the fox moves either one hole to the right, or one to the left (if it is still alive). From hole 1 it can only go to hole 2, and from hole 5, it can only go to hole 4. Prove that you can win in a finite number of turns.