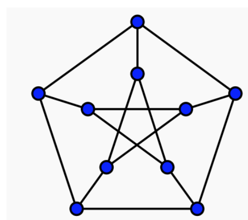
Graph theory - problem set 5

March 23, 2017

Exercises

1. For a graph G, we define its *line graph* $L(G)$ as follows: $V(L(G)) = E(G)$, $E(L(G)) = \{\{e, e'\} : e, e' \in V(L(G)), e \cap e' \neq \emptyset\}$. In other words, the vertices of $L(G)$ correspond to the edges of G and two such edges are adjacent in $L(G)$ they share a common endpoint in G.
   (a) Draw the line graph of K_4.
   (b) Find a graph G such that the line graph of G is K_n.
2. What is the vertex connectivity of K_4? What is the edge connectivity of K_4?
3. Calculate the vertex and edge connectivity of the following graph. Then choose a vertex and delete it. Calculate the new edge and vertex connectivity. Does it depend on the vertex you deleted?



4. Prove that any minimal edge separator is an edge cut.
5. Prove the following variants of Menger's theorem. Let G be a graph and let S, T be disjoint vertex sets. An *S-T path* is a path with one endpoint in S and the other in T. Then:
   (a) The maximum number of edge-disjoint S-T paths equals the minimum size of an S-T edge cut.
   (b) If $|S|, |T| \geq k$ and there is no S-T separator of size k, then G contains k vertex disjoint S-T paths. (An S-T separator $X \subseteq V(G)$ is a set such that $G - X$ has no path between $S \setminus X$ and $T \setminus X$.)

Problems

6. Find a graph G with $\kappa(G) = 10$ and $\kappa'(G) \geq 100$.
7. Let G be a graph and u, v be vertices in G. Show that a u-v vertex separator X is minimal (i.e. there is no proper subset $Y \subsetneq X$ that separates u and v) if and only if every vertex in X has a neighbor in the component of $G - X$ containing u and another in the component containing v.
8. Show that if G is a graph with $|V(G)| = n \geq k + 1$ and $\delta(G) \geq (n + k - 2)/2$ then G is k-connected.
9. Deduce the global version of Menger's theorem for *edges* from the global version for *vertices*.
   [Hint: Use the line graph defined in Exercise 1.]
10. A *k-factor* in G is a k-regular spanning subgraph of G (e.g., a 1-factor is a perfect matching). Let G be a bipartite graph with parts $A \cup B$ such that $|A| = |B| = n$. For a set $X \subseteq A$, let $N_i(X) \subseteq B$ be the set of vertices that have at least i neighbors in X.
    (a) Prove that if $|N_1(X)| + |N_2(X)| \geq 2|X|$ for every set $X \subseteq A$ then G has a 2-factor.
    (b) More generally, prove that if $\sum_{i=1}^{k} |N_i(X)| \geq k|X|$ for every set $X \subseteq A$ then G has a k-factor.
11. * Show, without using Menger's theorem, that if G is a 2-connected graph, then for every two vertices in G there is a cycle containing both vertices.