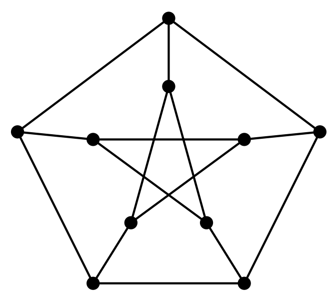
Graph theory - problem set 4

March 15, 2017

Exercises

1. In this exercise we show that the sufficient conditions for Hamiltonicity that we saw in the lecture are "tight" in some sense.
   (a) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has $\binom{n-1}{2} + 1$ edges.
   (b) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has minimum degree $\lceil \frac{n}{2} \rceil - 1$.
   (c) For every $k, n \geq 2$, find a graph G on *at least* n vertices such that $\delta(G) = k$ but G contains no cycle longer than $k + 1$.
2. Check that the proof of Dirac's Theorem also proves the following statement (called Ore's theorem): If for all non-adjacent vertices u, v in an n-vertex graph G we have $d(u) + d(v) \geq n$, then G has a Hamilton cycle.
3. The graph below is called the Petersen graph. Does it have a Hamilton path? And a Hamilton cycle?



Problems

4. Use Ore's theorem to give a short proof of the fact that any n-vertex graph G with more than $\binom{n-1}{2} + 1$ has a Hamilton cycle.
5. Let G be a connected graph on n vertices with minimum degree δ. Show that
   (a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ, and
   (b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamiltonian path.
6. Prove that the only acyclic tournament (with no directed cycle) is the transitive tournament.
7. Prove that if a tournament contains a directed cycle (i.e., it is not the transitive tournament) then it contains a directed triangle (3-cycle), as well.
8. Suppose each edge of the complete graph K_n has either a red or a blue color. Prove that this colored graph has a Hamilton path that is the union of a red path and a blue path. (We allow the case when one of the paths has length 0, i.e., the Hamilton path uses only one color.)

9.* We say that a vertex u in a tournament is "almost central" if for every other vertex v, there is a directed u-v path of length at most 2. Prove that every tournament has an almost central vertex.