Graph Theory: Problem set 3

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1. **a)** Every 1-connected graph, which is not 2-connected, is a tree.

False, consider a cycle with an additional edge attached to it.

b) Every tree is 1-connected and not 2-connected.

True. Any neighbor of a leaf is a cut vertex, if the tree has at least three vertices. Possibly a tree consisting only of one edge could be a counterexample. However, in general if a graph has less than k+1 vertices, k-connectedness is usually not defined.

c) No k-connected graph can contain a vertex of degree $k-1$.

True. The neighbors of a vertex v of degree $k-1$ separate it from the rest of the graph. Again, we do not define k-connectedness for a graph on less than $k+1$ vertices.

d) Every graph with minimal vertex degree k is k-connected.

False. A cut-vertex can have an arbitrary degree.

2. Prove without using Menger's theorem that in a 2-connected graph every two vertices lie on a common cycle.

By using the fact that every 2-connected graph can be constructed by successively adding H-paths to a cycle we prove our claim by induction on the number of edges in G. Thus, the trivial case is when whole G is a cycle, otherwise we can assume that we can obtain G by adding an H-path wPz to a proper 2-connected subgraph H of G. If we pick two arbitrary vertices u and v from $V(G)$, either both of them belong to H, both of them belong to $P \setminus H$, or one of them belongs to H, let's say u, and the other to $P \setminus H$. The first case can be handled by induction hypothesis. For the second case it is enough to use the fact that H is connected. Indeed, a path wP_0z within H can be extended with wPz to the cycle containing u and v. For the third case we show that there is a path from w to z via u. By induction hypothesis w and u lie on a common cycle C_1 and z and u lie on a common cycle C_2. If $z \in C_1$ we are done. Otherwise let f be the first vertex belonging to C_2 on a path P_1 from w to u that takes place in C_1. f is well defined because at least u belongs to C_2. Let fP_2z be the path that takes place in C_2, and contains u (see Figure

1 for an illustration). By the definition of f wP_1fP_2z is the path from w to z containing u. Putting wP_1fP_2z and wPz gives us the cycle containing u and v.

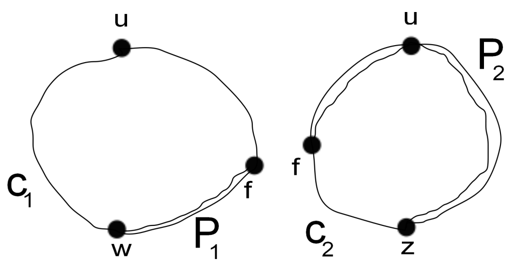

Figure 1: The path P_1P_2

3. Prove that any k-regular connected bipartite graph is 2-connected.

We can assume that $k > 1$, as other cases are trivial. We proceed by contradiction. Let v be a cutvertex in a k-regular connected bipartite graph G with parts V_1 and V_2. Let $v \in V_1$. Let $C_1 \dots C_i$ denote the connected components of $G \setminus v$. Let k' denote the number of neighbors of v in C_1. If we count the edges in C_1 by looking at the vertices in $V(C_1) \cap V_1$ we get that k **divides** $|E(C_1)|$, because every vertex in $V(C_1) \cap V_1$ has degree k. On the other hand if we count the edges in C_1 by looking at the vertices in $V(C_1) \cap V_2$, we get that k does not divide $|E(C_1)|$. Indeed, all vertices in $V(C_1) \cap V_2$, but k', where $0 < k' < k$, have degree k, and k' vertices has degree $k - 1$. Hence, in total the number of edges $k'(k-1) + lk = k(k'+l) - k'$, for some l, cannot be divisible by k (contradiction).

4. Using Dilworth theorem prove that any sequence of real numbers of length $n^2 + 1$ contains a monotone increasing or a monotone decreasing subsequence of length $n + 1$.

Let $S = (x_1, \dots x_{n^2+1})$ denote a sequence of real numbers. We associate with the element x_i the point $X_i = (i, x_i)$ in the plane. Let us define the relation $\leq_r$ on the set of points $X = \{X_i \mid i = 1, .., n^2 + 1\}$ as follows: $(a, b) \leq_r (c, d)$ iff $a \leq c$ and $b \leq d$. It is straightforward to check that $(X, \leq_r)$ is a poset.

Let m denote the length of a longest chain in X.

If $m \geq n + 1$, we are done, since a chain corresponds to the monotone increasing subsequence of S.

If $m < n + 1$, by Dilworth theorem we can partition X into at most m anti-chains. If the length of each of these anti-chains is less than $n + 1$, we obtain a contradiction. Indeed, we partition X into at most n anti-chains each of which has at most n elements. Thus, we could have at most $n * n$ elements in S. However, the size of S is $n^2 + 1$. Hence, we can find an anti-chain of size $n + 1$, which corresponds to the monotone decreasing subsequence of S, and that concludes the proof.

5. $*$ Prove marriage (Hall) theorem by using Tutte's theorem.

Suppose that Hall theorem holds for bipartite graphs having the two parts of the vertices of the same size.

Let $G = (V = A \cup B, E)$ denote a bipartite graph with bipartition A and B, so that $A \leq B$. Suppose that marriage condition holds for A i.e. $|N(A')| \geq A'$, for any $A' \subseteq A$. First, we reduce the problem to the situation when $|A| = |B|$. This can be done by adding dummy vertices to A and connect each of them with every vertex in B.

In what follows we show that if $|A| = |B|$ and the marriage condition is satisfied, then for G the condition of Tutte's theorem holds as well. Let $S \subseteq V$ denote a non-empty subset of V. Let $S_A = A \cap S$. Let c_B denote the number of connected components of $G \setminus S$ having more vertices of B than A. Analogously, we define c_A.

We claim that $c_B \leq |S_A|$. Indeed, let B' denote the subset of B consisting of the vertices in the union of these connected components. By marriage condition we have at most $|S_A|$ of such components, since the number of neighbors of the vertices in B' in $S \setminus S_A$ is at most $|B'| - c_B$. Similarly, we prove that $c_A \leq |S_B|$.

Let c denote the number of odd connected components in $G \setminus S$. Obviously, $c_A + c_B = c$. Thus, $c_A + c_B = c \leq |S|$. Hence, the Tutte's condition is satisfied, and marriage theorem follows.

6. Hint: Define a partial order on the vertices by directing all edges from X to Y for $V = X \cup Y$. By Dilworth's theorem, one can partition the vertices into k chains, such that the size of the largest antichain A is k. Argue that the number of chains of length two (in our partition) equals the size of the complement of A.