Graph Theory: Problem set 2

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1. Let us have a system of subsets $\mathcal{S}$ of a finite set $\{1, 2, \dots n\}$ such that for any $0 < k \leq |\mathcal{S}|$ the following statement is true: Any union of k sets from $\mathcal{S}$ contains at least k elements. Prove that one can choose for each set S in $\mathcal{S}$ a representative $r = r(S) \in S$ so that no element of $\{1, 2, \dots n\}$ is a representative for two sets of $\mathcal{S}$, i.e. $r(S_1) \neq r(S_2)$ if $S_1 \neq S_2$.

Define a bipartite graph $G = (V_1 \cup V_2)$ with parts of size $|\mathcal{S}|$ and n, respectively. Each vertex in the first part V_1 corresponds to an element (a set) of $\mathcal{S}$ and each vertex in the second part V_2 corresponds to an element in $\{1, 2, \dots n\}$. Two vertices in G are joined by an edge iff they correspond to a set S and an element e such that $e \in S$, respectively. Now, we can apply Hall's Theorem on G. Notice that by our assumption any subset V' of V_1 is joined to at least $|V'|$ vertices in V_2. Indeed, there are at least $|V'|$ elements contained in the union of the sets corresponding to V'. Thus, by Hall's Theorem there is matching M that matches every vertex in V_1 with a vertex in V_2. We choose our representatives as follows. For a set $S \in \mathcal{S}$, whose corresponding vertex in V_1 was matched by M with a vertex $v \in V_2$, we choose the element v corresponds to.

2. We define the *complement* G^c of a graph G as the graph on the same vertex set in which two vertices are joined by an edge if and only if they are not joined by an edge in G.

Prove that it cannot happen that both G and G^c are disconnected.

Let $C_1, \dots C_k$ denote the connected components of G.

We can assume that $k > 1$, otherwise there is nothing to prove, since in this case G is connected. Thus, by definition of the connectedness, it is enough to find a path between any two vertices in G^c. In the complement of G between any two vertices belonging to the different connected components of G there is an edge in G^c. Moreover, if u and v belong to the same connected component C_i, there is a path of length 2 between u and v via a vertex w belonging to another connected component C_j, $i \neq j$.

3. Let $\alpha(G)$ denote the *independence number* of G, that is, the maximum number of vertices that can be chosen so that no two of them are connected by an edge. Show that the vertex set of every graph G can be covered by at most $\alpha(G)$ *vertex disjoint* paths (that is, paths that do not share any vertices).

Let $\mathcal{P}$ be a path cover of G with minimal number of paths. Let us consider the subset V' of the vertices of G consisting only of the endpoints of the paths in $\mathcal{P}$ (for each path we put into V' exactly one endpoint). V' is an independent set as otherwise we have two vertices u, v in V' joined by an edge, which allows us to join their respective paths $P_u u$ and vP_v into a path $P_u uvP_v$ giving us a smaller path cover and thereby contradicting our assumption.

4. We define the diameter $\mathrm{diam}(G)$ of a graph G to be the maximum distance at which two vertices in G may be from each other. We define the radius $\mathrm{rad}(G)$ of a graph G to be the minimum over all vertices of G of the maximum distance of a vertex v in G to some other vertex, i.e. $\min_{v\in V(G)} \max_{u\in V(G)} \mathrm{dist}(u, v)$. Show that $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\mathrm{rad}(G)$.

By definition we have $\mathrm{rad}(G) \le \mathrm{diam}(G)$, as $\mathrm{rad}(G)$ is a distance between some pair of vertices in G, which cannot be bigger than the maximum distance in G. Let v be a vertex minimizing $\max_{u\in V(G)} \mathrm{dist}(u, v)$. We have a path of length at most $\mathrm{rad}(G)$ from v to any vertex of G. Thus, if u and w are two vertices in G there must exist a walk of length at most $2\mathrm{rad}(G)$ joining them through v. Hence, there exists a path of length at most $2\mathrm{rad}(G)$ joining them.

5. We call the cube the graph $G = G(V, E)$ whose vertex set is the set $\{0, 1\}^n$ of all n-dimensional $0 - 1$ vectors and in which two vertices form an edge if their corresponding vectors differ in exactly one component. Show that the cube is bipartite.

The desired partition of the vertex set of G into two parts witnessing its bipartiteness looks as follows.
$V_1 = \{u \in \{0, 1\}^n |\ u \text{ contains odd number of ones}\}$ and $V_2 = V \setminus V_1$.

We show that there is no edge within V_1 or V_2. If $uv \in E(G)$, and the number of ones in u is odd, then the number of ones in v is even (by the definition of $E(G)$). If $uv \in E(G)$, and the number of ones in u is even, then the number of ones in v is odd (by the definition of $E(G)$).

6. Bonus hint: One can two color the checkerboard so that every 2-by-1 and 1-by-2 rectangle contains both colors, and the opposite corners have the same color.