Graph Theory: Problem set 2

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1. Let us have a system of subsets $\mathcal{S}$ of a finite set $\{1, 2, \dots n\}$ such that for any $0 < k \le |\mathcal{S}|$ the following statement is true: Any union of k sets from $\mathcal{S}$ contains at least k elements. Prove that one can choose for each set S in $\mathcal{S}$ a representative $r = r(S) \in S$ so that no element of $\{1, 2, \dots n\}$ is a representative for two sets of $\mathcal{S}$, i.e. $r(S_1) \ne r(S_2)$ if $S_1 \ne S_2$.

2. We define the *complement* G^c of a graph G as the graph on the same vertex set in which two vertices are joined by an edge if and only if they are not joined by an edge in G.

Prove that it cannot happen that both G and G^c are disconnected.

3. We define the diameter $\mathrm{diam}(G)$ of a graph G to be the maximum distance at which two vertices in G may be from each other. We define the radius $\mathrm{rad}(G)$ of a graph G to be the minimum over all vertices of G of the maximum distance of a vertex v in G to some other vertex, i.e. $\min_{v \in V(G)} \max_{u \in V(G)} \mathrm{dist}(u, v)$. Show that $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\mathrm{rad}(G)$.

4. We call the cube the graph $G = G(V, E)$ whose vertex set is the set $\{0, 1\}^n$ of all n-dimensional $0 - 1$ vectors and in which two vertices form an edge if their corresponding vectors differ in exactly one component. Show that the cube is bipartite.

5. Show that a tree without a vertex of degree two (by degree of a vertex we mean the number of vertices it is joined with) contains more leaves (i.e. vertices of degree one) than non-leaves.

6. $*$ Let $\alpha(G)$ denote the *independence number* of G, that is, the maximum number of vertices that can be chosen so that no two of them are connected by an edge. Show that the vertex set of every graph G can be covered by at most $\alpha(G)$ *vertex disjoint* paths (that is, paths that do not share any vertices).