Graph theory - problem set 11

May 17, 2018

Exercises

1. Let $(\Omega, \mathbb{P})$ be a probability space. Prove that for any collection of events $E_1, \dots, E_k$, we have

$$\mathbb{P}\left[\bigcup_{i=1}^{k} E_i\right] \leq \sum_{i=1}^{k} \mathbb{P}[E_i],$$

   and if $E_1, \dots, E_k$ are disjoint events, then we have equality here.

2. Let σ be an arbitrary permutation of $\{1, \dots, n\}$, selected uniformly at random from the set of all permutations (that is, each permutation is selected with probability $\frac{1}{n!}$). What is the expectation of the number of fixed points in σ? (Recall that i is a fixed point if $\sigma(i) = i$.)

3. Take a complete graph K_n where each edge is independently colored red, green or blue with probability 1/3. What is the expected number of red cliques of size a in this graph?

Problems

4. Suppose $r \geq 4$ and let H be an r-uniform hypergraph with at most $4^{r-1}/3^r$ edges. Prove that there is a coloring of the vertices of H by four colors so that in every edge all four colors appear.

5. Let G be a graph with m edges, and let k be a positive integer. Prove that the vertices of G can be colored with k colors in such a way that there are at most m/k monochromatic edges (i.e., edges with both endpoints colored the same).

6. Prove that if G has $2n$ vertices and e edges then it contains a bipartite subgraph with at least $e\frac{n}{2n-1}$ edges. [Use a random partition of the vertices into two parts of size n]

7. (a) Let G be a $K_{1,d}$-saturated graph. Prove that the vertices of G whose degrees are smaller than $d-1$ form a clique.

   (b) Use this to prove that $\mathrm{sat}(n, K_{1,d}) \geq \frac{n(d-1)}{2} - \frac{d^2}{8}$.

   (c) Prove that $\mathrm{sat}(n, K_{1,d}) = \frac{n(d-1)}{2} - \frac{d^2}{8}$ if d is even and $n - d/2$ is also even.

   [This problem is not probabilistic]

8.* In an $n \times n$ matrix, each of the numbers $1, 2, \dots, n$ appears exactly n times. Show that there is a row or a column in the matrix with at least $\sqrt{n}$ distinct numbers.