Graph Theory: Problem set 11

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1. Show that the only graph on n vertices without a triangle, that has $\lfloor n^2/4 \rfloor$ edges, is the complete balanced bipartite graph.

Let G denote a triangle free graph on n vertices and with $\lfloor n^2/4 \rfloor$ edges that is not the complete balanced bipartite graph. By Mantel's Theorem any triangle-free graph G' on n vertices contains at most $\alpha(G')\tau(G')$ edges. Since G maximize the number of edges, we have that $\tau(G) + 1 \geq \alpha(G) \geq \tau(G) - 1$.

G has $\lfloor n^2/4 \rfloor$ edges and an edge uv between two vertices in a smallest vertex cover V' of G. Since G is triangle free, the neighbourhoods (the sets of neighbours of u and v) of u and v are disjoint.

Observe that the number of vertices in $V \setminus V'$ connected to at least one vertex of the neighbourhood $N_v \subseteq V$ of v, for some $v \in V'$, is at least the size of N_v. Indeed, otherwise we could increase the size of our maximal independent set $V \setminus V'$ by replacing in $V \setminus V'$ all neighbours of the vertices in N_v with N_v (contradiction).

In the light of the above observations we can delete all edges within V' incident to $v \in V'$, and join v with all vertices in $V \setminus V'$, v was not connected to before, i.e. to at least $|N_v|$ vertices (see Figure **??** for an illustration).

Since the number of edges cannot decrease by this operation, we can get rid of all edges within V', and obtain a triangle free bipartite graph G' with at least as many edges as G has. Moreover, if u is the last vertex whose incident edges are removed, we can add at least one additional edge between V' and $V \setminus V'$. That is easy to see, because u has, at this stage, at least one neighbour in V', which is of course of degree 1 (at this stage). Hence, after we remove the edge uv, we can add an edge vu', such that $u'u \in E$, and $u' \in V \setminus V'$. Notice, that u has to be connected to at least one vertex in $V \setminus V'$, because otherwise we could increase the size of a biggest independent set.

Hence, we obtain the complete balanced bipartite graph on n vertices having more edges than G (contradiction).

2. For $0 < s \leq t \leq n$ let $z(n, s, t)$ denote the maximum number of edges in a bipartite graph whose partition sets both have size n, and which does not contain $K_{s,t}$. Show that $2ex(n, K_{s,t}) \leq z(n, s, t) \leq ex(2n, K_{s,t})$.

Suppose the second inequality does not hold. It follows that there is a bipartite on $2n$ vertices having more than $ex(2n, K_{s,t})$ edges and not containing $K_{s,t}$ (contradiction).

Let $G = (V, E)$ denote a graph on n vertices with $ex(n, K_{s,t})$ edges which does not contain $K_{s,t}$. We construct the bipartite graph $G' = (V_1 \cup V_2, E')$ on $2n$ vertices with the two parts V_1 and V_2, s.t. $V_i = \{u_i |\ u \in V\}$, of equal size and the set of edges $E' = \{u_1v_2, u_2v_1 |\ u_i, v_i \in V_i,\ i = 1, 2,\ uv \in V\}$. Clearly, $|E'| = 2|E|$. Moreover, G' does not contain $K_{s,t}$. Indeed, any copy of $K_{s,t}$ in G' would give us a copy of $K_{s,t}$ in G (by discarding the indices at the vertices). Thus, $2ex(n, K_{s,t}) \le z(n, s, t)$.

3. Let G denote a graph on n vertices with chromatic number k. At most how many edges can G have ?

By Turán's Theorem any graph on n vertices with more than
$T_k(n) = \frac{1}{2}((n \mod k) \left\lceil \frac{n}{k} \right\rceil (n - \left\lceil \frac{n}{k} \right\rceil) + (n - (n \mod k)) \left\lfloor \frac{n}{k} \right\rfloor (n - \left\lfloor \frac{n}{k} \right\rfloor)) \approx \frac{k-1}{2k} n^2$
edges contains K_{k+1}, which has chromatic number $k + 1$. Hence, we cannot hope that the answer is bigger than $T_k(n)$.

On the other hand the complete balanced k partite graph, which has exactly $T_k(n)$ edges has chromatic number k.

Thus, the answer is exactly $T_k(n)$.

4. Let $H = (V, E)$, $n' = |V|$, denote a graph on at least 2 vertices. We define $ex_H(n)$ as the maximal number of edges a graph on n vertices can have, if it does not contain H as a subgraph.

Prove that the maximal number of edges a graph on n vertices can have, if it does not contain two disjoint copies of H as a subgraph, is at most $ex_H(n) + nn' + c_H$, where c_H is a constant depending only on the number of vertices in H.

Let G denote a graph on n vertices with at least $ex_H(n) + nn' + c_H + 1$ edges (n' and c_H are specified later). We show that G contains two disjoint copies of H. That clearly proves the claim. Clearly, G contains H as a subgraph. Let G' denote a subgraph of G isomorphic to H. By removing G' from G together with all the edges incident to any vertex in G', we remove at most $(n - |V(H)|)|V(H)| + \binom{|V(H)|}{2}$ edges. By setting $n' = |V(H)|$ and $c_H = \binom{|V(H)|}{2} - |V(H)|^2$, at least $ex_H(n) + 1$ edges in G still remain, which yields another copy of H in G disjoint from G'.

5. Let T denote a tree on 4 vertices. Prove the if G has minimum degree 3, then T is a subgraph of G.

Let us prove a stronger claim. Let G denote a graph with each vertex of degree at least $t - 1$. Then G contains any tree on t vertices as a subgraph. We proceed by induction on t. The base case for $t = 2$ is easy.

For $t > 2$, let $T' = T \setminus v$, where v is a leaf. By our induction hypothesis G contains T' as a subgraph G' Let u be the vertex of G' corresponding to the only neighbour of v in T. Since the degree of u is at least $t-1$, u has still a neighbour v' not belonging to G', as G' has $t-1$ vertices. By adding uv' to G' we obtain a subgraph isomorphic to T in G.

6. $*$ Let n, m, s and t denote four natural numbers such that $0 < s < n$ and $0 < t < m$. Show that by deleting at most $(n-s)(m-t)/s$ edges from $K_{n,m}$ we cannot destroy all its $K_{s,t}$ subgraphs.

Let A, B denote the two parts of vertices in $K_{n,m}$ of size n and m, respectively. Notice that as long as we have in A at least s vertices each of which has degree at least $m-(m-t)/s$ we can find $K_{s,t}$ subgraph. Really, each of these at least s vertices is not connected to at most $(m-t)/s$ vertices in B. Hence, we can choose s vertices in A that are all connected to a t-tuple of vertices in B.

If we delete from $K_{s,t}$ at most $(n-s)(m-t)/s$ edges it follows that there are still at least s vertices in A each of which has degree at least $m-(m-t)/s$. Indeed, otherwise we have at least $n-s+1$ vertices in A each of which has the degree at most $m-(m-t)/s-1$. Thus, the total number of edges in $K_{n,m}$ is at most $(n-s+1)(m-(m-t)/s-1)+(n-s)(m-t)/s+(s-1)m = (n-s)(m-1)+m-(m-t)/s-1+(s-1)m = nm-n+s-(m-t)/s < mn$ (contradiction).