Graph theory - problem set 11

May 11, 2017

Exercises

1. Show that any graph on $R(s,t)$ vertices contains a clique of size s or an independent set of size t.

2. Let G be a graph on 6 vertices such that $\alpha(G) < 3$. Prove that G contains a triangle.

3. Determine $R(P_3, K_3)$ and $R(P_4, P_4)$ (recall that P_k is a path on k vertices).

4. The lower bound for $R(p,p)$ that we saw in the lecture is not a constructive proof: it merely shows the *existence* of a red-blue coloring not containing any monochromatic copy of K_p by bounding the number of bad graphs. Give an explicit coloring on $K_{(p-1)^2}$ that proves $R(p,p) > (p-1)^2$.

5. Prove that $R(n_1, \ldots, n_k) \leq R(n_1, \ldots, n_{k-2}, R(n_{k-1}, n_k))$. Deduce that for every k and n, there is an N such that any k-coloring of the edges of K_N contains a monochromatic K_n.

Problems

6. (a) Prove that $R(4,3) \leq 10$, i.e., any graph on 10 vertices contains a clique of size 4 or an independent set of size 3.

   (b) Prove that $R(4,3) \leq 9$.

7. Show that any 2-coloring of the edges of K_6 contains at least two monochromatic triangles.

8. Prove that for every $k \geq 2$ there exists an integer N such that every coloring of $[N] = \{1, \ldots, N\}$ with k colors contains three numbers a, b, c satisfying $ab = c$ that have the same color.

9. The edges of K_n with $n \geq 3$ are colored with two colors. Prove that it contains a Hamiltonian cycle that is the union of two monochromatic paths.

10. *Prove that for every fixed positive integer r, there is an n such that any coloring of all the subsets of $[n]$ using r colors contains two non-empty disjoint sets X and Y such that X, Y and $X \cup Y$ have the same color.