Graph Theory: Problem set 10

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1. At most how many edges must a graph have, if it does not contain an odd cycle as a subgraph ?

If a graph G does not contain an odd cycle, then G must be a bipartite graph. Thus, we can partition the vertex set V of G into two parts V_1 and V_2, i.e. $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that every edge in G joins a vertex in V_1 with a vertex in V_2. The maximum number of edges, G can have, is $|V_1||V_2|$, which corresponds to the case when every vertex in V_1 is connected to every vertex in V_2. If $|V| = n$, the above expression becomes $f(|V_1|) = |V_1|(n-|V_1|)$. The maximum of the previous function is in $n/2$. Indeed, the first derivation of f is $f'(|V_1|) = n - 2|V_1|$. Thus, the answer is $n/2(n - n/2) = n^2/4$.

2. At least how many edges must a graph on n vertices have, if it does not contain an independent set of size 3 ?

A graph G has an independent set of size 3 if and only if its complement $\overline{G}$ contains a clique of size 3.

A graph G has at most m edges if and only if is complement $\overline{G}$ has at least $\binom{n}{2} - m$ edges.

Moreover, by Turán's Theorem we know that a graph on n vertices, which does not contain a clique of size 3, has at most $\left\lfloor \frac{n^2}{4} \right\rfloor \approx \frac{n^2}{4}$ edges.

Thus, if G does not have an independent set of size 3, its complement $\overline{G}$ does not contain a clique of size 3. Hence, $\overline{G}$ has at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, and therefore its complement $G = \overline{\overline{G}}$ has at least $\binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor \approx \frac{n^2}{4}$ edges.

3. $*$ At most how many edges must a graph on n vertices have, if all of its cycles share at least one vertex with one cycle C of length 5 ? (Show the best upper bound you can.)

Let G be a graph that satisfies our condition. If G contains a cycle of length 5, by its removal (i.e. by removal of all of its vertices with all their incident edges) we remove at most $5(n - 5) + 10 = 5n - 15$ edges. As the remaining part of G is a tree, G does not contain more than $5n - 15 + n - 6 = 6n - 21$ edges. Otherwise G is a tree, and thus, it contains at most $n - 1$ edges.

4. Let G be a graph containing at least $\frac{3}{2}(n-1)+1$ edges. Prove that G contains two vertices joined by three internally pairwise disjoint paths.

The crucial observation is as follows.

If we add to $E(G)$ an edge $e \notin E(G)$ joining two vertices u and v (in the same connected component) of G, such that there are at least two different paths between u and v, then we obtain three internally pairwise disjoint paths in G.

Indeed, let $uP_1v = uP_1u'P_1v'P_1v$ and $uP_2v = uP_2u'P_2v'P_2v$ denote two different paths between u, and v. We have chosen u' and v' such that $u'P_1v'P_2u'$ is a cycle. Formally, u' is defined as the vertex belonging to both P_1 and P_2 such that the vertices following u' on P_1 and P_2, resp., are different, and v' is defined as the first vertex on P_1 after u' that belongs to P_2. If we have a vertex other than u' and v' on both $u'P_1v'$ and $v'P_2u'$, the definition of v' is violated. Thus, we have three internally pairwise disjoint paths between u' and v', Namely, $u'P_1v'$, $u'P_2v'$ and $u'P_1vuP_2v'$.

We proceed with the proof of the claim using the observation from above. First let us handle the case when G is connected. $G=(V,E)$ has a spanning tree $T=(V,E')$ with $n-1$ edges. Let us order the edges in $E''=E\setminus E'=\{e_1,e_2\dots e_k\}$, and let $G_i=(V,E'\cup\{e_1,\dots e_i\})$, and $G_0=G$. Since none of G_i, $1\le i\le k$, contains two vertices joined by three internally pairwise disjoint paths, for each $e_i=v_iu_i$, $1\le i\le k$, there is only one path between u_i and v_i in G_{i-1}. That also means that every vertex on the path u_iPv_i in G_{i-1} is a cut vertex. Indeed, if after removing an internal vertex from u_iwPv_i, we have u_i and v_i in the same connected component, there is another path joining them. If u_i is not a cut vertex, u_i is contained in a unique 2-connected block and we have a path from u_i to w different from u_iw, that can be used to construct a different path than P from u_i to v_i. Hence, no two e_i and e_j, $i<j$, are contained in the same block, because a block in G_j containing e_j is in fact a single cycle. That is easy to see, because the cycle $u_jPv_ju_j$ does not contain a chord, and if there is a vertex in the same block as e_j outside this cycle, we could construct another path between u_j and v_j in G_{j-1}.

Thus, each block of G is either an edge or a cycle.

Now, we can continue by induction on the number of blocks in the block decomposition of G.

If G has only one block, we are done, because the worst case is that G is a cycle. Thus, G contains at most n edges.

If G has more than one block, one of them B (with $|B|>2$ vertices) is a leaf in the block decomposition tree of G. If we remove B from G without removing its cut vertex (it has only one cut vertex), we remove at most $|B|$ edges, and exactly $|B|-1$ vertices from G . Thus, using induction hypothesis the number of vertices in $|V(G)|$ is at most $|E(G)|\le\frac{3}{2}(n-(|B|-1)-1)+|B|=\frac{3}{2}(n-1)-\frac{1}{2}|B|+\frac{3}{2}$. If $|B|\ge 3$ the claim follows immediately, otherwise we could remove at most one edge and we have $|E(G)|\le\frac{3}{2}(n-(|B|-1)-1)+1\le\frac{3}{2}(n-(2-1)-1)+1\le\frac{3}{2}(n-1)$

If G is not connected we can proceed by the induction on the number of connected components.

The base case is now trivial, since it was treated above. If G consists of k connected components, let G_1 denote a subgraph of G consisting of $k-1$ connected components of G, and let G_2 denote remaining connected component. Using induction hypothesis we obtain $|E(G)| = |E(G_1)| + |E(G_2)| \leq \frac{3}{2}(|V(G_1)| - 1) + \frac{3}{2}(|V(G_2)| - 1) \leq \frac{3}{2}(n-1)$

5. Let V denote a finite set of points on a unit circle S^1 in the plane. By a unit circle we understand the set $S^1 = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1\}$.

a) What is the maximum number of unordered pairs $u, v \in V$ such that $|uv| \geq 1.8$?

Let $G = (V, E)$ denote the graph whose edge set is defined as follows.
$uv \in E$ if and only if $|uv| \geq 1.8$. G is triangle free, because the size of the convex angle $u0v$, $uv \in E$, is at least 143 degrees. Thus, G has at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, which in turn implies that the maximum number of pairs in V at distance at least 1.8 is at most $\left\lfloor \frac{n^2}{4} \right\rfloor$.
On the other hand the above upper bound is the best one, as witnessed by the following construction. Let $V = V_1 \cup V_2$, where V_1 consists of $\lfloor \frac{n}{2} \rfloor$ points very close to $(0, 1)$ and V_2 consists of $\lceil \frac{n}{2} \rceil$ points very close to $(0, -1)$.

b) What is the maximum number of unordered pairs $u, v \in V$ such that $|uv| \geq 1.5$?

We can proceed along the same lines as in the part a) except, that now, we are not allowed to have K_4 in G. Really, if we have a quadrilateral $uvwz$, such that $u, v, w, z \in V$, with the diagonals uw and vz, and z belonging to a shorter arc on S^1 defined by vw, either uz or wz is shorter than $\sqrt{2} < 1.5$.
Again by Turán's Theorem we obtain the tight bound, as witnessed by the construction, which consists of three set of points each of which consists of the points that are very close to a point of an equilateral triangle with its three vertices on S^1, respectively.

6. Let G be a graph, in which each vertex has a degree at least 3. Prove that G contains a cycle with a chord, i.e. an edge joining two non-consecutive vertices on the cycle. (*hint: a longest path*)

Notice that a longest path in G must have at least three vertices, since two neighbors b, c of any vertex a form a path bac. Let $uwPv$ denote a longest path in G. As every vertex in G has a degree at least 3, we have two neighbors n_1 and n_2 of u not equal to w. Since P is a longest path, $n_1, n_2 \in P$. Indeed, otherwise we would obtain a path longer than P. Namely, n_1uwPv or n_2uwPv. The cycle in G with a chord is either uPn_2u or uPn_1u, depending on whether n_1 occurs on P before n_2, or after n_1. In the first case the chord is un_1 and in the second case the chord is un_2.