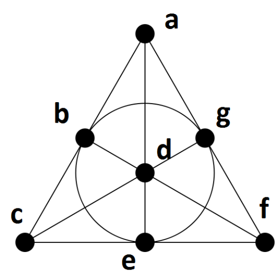
Discrete mathematics - problem set 9

November 13, 2014.

1. The Fano plane (represented in the picture below) is a triple system consisting of the following 7 triples: $\{a,b,c\},\{a,d,e\},\{a,f,g\},\{b,d,f\},\{b,e,g\},\{c,d,g\},\{c,e,f\}$.



   Prove that the Fano plane is a maximal intersecting family of 3-sets, that is no other subset of three points could be added such that the family remains intersecting. Compare with the maximum in the Erdős-Ko-Rado theorem.

2. Given a set of 15 elements. Prove that the maximum cardinality of an intersecting family of 8-element sets is at least as large as the maximum cardinality of an intersecting family of 10-element sets.

3. Let $k \geq n/2$ and let $\mathcal{F}$ be a family of k-element subsets of $\{1,2,...,n\}$ such that $A \cup B \neq \{1,2,...,n\}$, for all $A, B \in \mathcal{F}$. Prove that $|\mathcal{F}| \leq \binom{n-1}{k}$. Provide an example of a family $\mathcal{F}$ of size exactly $\binom{n-1}{k}$.

4. Let $k = n/2$, with n even. Prove that there are exactly $2^{\binom{n-1}{k-1}}$ families $\mathcal{F}$ of k-elements subsets of $\{1,2,...,n\}$ of size $|\mathcal{F}| = \binom{n-1}{k-1}$, such that any two members of $\mathcal{F}$ have non-empty intersection.

5*. In a factory producing sports equipment, there are 20 machines creating weights of 10 kilograms for gym. The owner of the factory knows that one of the machines is not working properly, that is, instead of producing weights of 10 kilograms, it produces weights of only 9 kilograms. There is a scale in his factory (which tells you the exact weight), but its battery is low, so one can perform only one weighing. Could you help the owner determine which machine produces lighter weights?