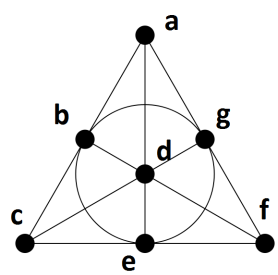
Discrete mathematics - problem set 8

November 5, 2015.

1. The Fano plane (represented in the picture below) is a triple system consisting of the following 7 triples: $\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{b, e, g\}, \{c, d, g\}, \{c, e, f\}$. In the figure, the points represent the elements of the base set, while the lines and the circle represent members of the set family (thus, we have 7 sets).



   Prove that the Fano plane is a maximal intersecting family of 3-sets, that is no other subset of three points could be added such that the family remains intersecting. Compare with the maximum in the Erdős-Ko-Rado theorem.

2. Given a set of 15 elements. Prove that the maximum cardinality of an intersecting family of 8-sets is at least as large as the maximum cardinality of an intersecting family of 10-sets.

3. Let $k \geq n/2$ and let $\mathcal{F}$ be a family of k-element subsets of $\{1, 2, ..., n\}$ such that $A \cup B \neq \{1, 2, ..., n\}$, for all $A, B \in \mathcal{F}$. Prove that $|\mathcal{F}| \leq \binom{n-1}{k}$. Provide an example of a family $\mathcal{F}$ of size exactly $\binom{n-1}{k}$ for exercise 1.

4. Let $k = n/2$, with n even. Prove that there are exactly $2^{\binom{n-1}{k-1}}$ families $\mathcal{F}$ of k-elements subsets of $\{1, 2, ..., n\}$ of size $|\mathcal{F}| = \binom{n-1}{k-1}$, such that any two members of $\mathcal{F}$ have non-empty intersection.

5. Let $\mathcal{F}$ be a family of k-sets of $\{1, \ldots, n\}$, such that no two elements of $\mathcal{F}$ intersect in more than 1 element. Prove that $|\mathcal{F}| \leq \binom{n}{2}/\binom{k}{2}$.

6*. Let S be the union of k disjoint, closed intervals in the unit interval $[0, 1]$. Suppose S has the property that for every real number d in $[0, 1]$, there are two points in S at distance d. Prove that the sum of the lengths of the intervals in S is at least $1/k$.