Packing and covering - problem set 8

April 8, 2014.

1. Let $c_1, ..., c_n$ be a set of n points in the plane, so that no three of them are on a line. Show that in the corresponding Voronoi diagram, the unbounded cells are exactly the ones belonging to the vertices of the convex hull of the points $c_1, ..., c_n$.

2. Let $c_1, ..., c_n$ be a set of n points in the plane. Show that the corresponding Voronoi diagram has at most $2n - 5$ vertices and at most $3n - 6$ edges.

3. Let $c_1, ..., c_n$ be a set of n points in the plane. Let D_i be the Voronoi cell of c_i. Assume that no three points are on a line, and no four points are in the boundary of a circle. (Show that it implies that no four Voronoi cells meet at a point). Let T be the assigned Delaunay triangulation: T is the planar graph with vertex set $\{c_1, ..., c_n\}$ so that c_i and c_j are adjacent if and only if their corresponding Voronoi cells D_i and D_j are adjacent (i.e., if their boundaries intersect).

   (a) Show that T is a triangulation of $\mathrm{conv}(c_1, ..., c_n)$, that is, show that the only face of T that is not a triangle is the outer face.

   (b) Let $a, b, c \in \{c_1, ..., c_n\}$ be the vertices of a face of T and C be the circle circumscribed around a, b, c. Show that no point of $c_1, ..., c_n$ is in the interior of C.

   (c) Show that if the circle circumscribed around $a, b, c \in \{c_1, ..., c_n\}$ has no points of $c_1, ..., c_n$ in its interior, then the edges ab, bc, ca are in T.

4. Let $\mathcal{C}$ be a packing of convex discs in $\mathbb{R}^2$ and assume that the plane is subdivided into polygonal regions (regions bounded by simple close polygons) $S_1, S_2, ...$ so that every S_i intersects at most one $C_i \in \mathcal{C}$ in its interior, $A(C_i)/A(S_i) \leq \delta$, and the diameter of S_i is at most Δ (where Δ and δ are constants not depending on i). Prove that the upper density of $\mathcal{C}$ in R^2 is at most δ.

5. Show that in the proof of Rogers's theorem, the arrangement of the four translates of the trigonal convex disc in second case (when a translate of C intersects the region T) extends to a parallelogram lattice packing of the disc on the plane.

6*. Let C be a convex disc inside the unit circle centred at the origin. Assume that from every point p on the unit circle (so $|p| = 1$), C is seen at a right angle, that is, the two lines tangent to C passing through p are perpendicular. Prove that C is centrally symmetric. You may try to generalize the statement for other angles as well!