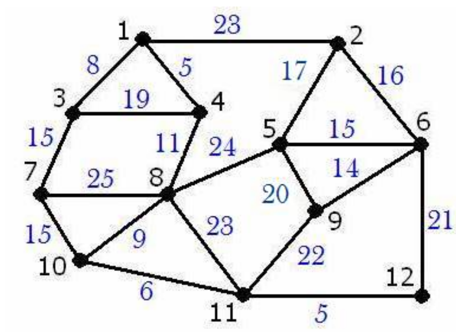
Discrete mathematics - problem set 5

October 15, 2015.

1. Prove that Kruskal's algorithm works correctly also in the case when two or more edge weights are equal, and the algorithm picks one of the lowest weight edges randomly.

2. Let V be the vertex set of a connected graph G, $U \subset V$, and e of minimum weight among the edges with one endpoint in U and the other endpoint in $V - U$. Prove that there exists a minimum spanning tree T such that $e \in T$.

3. Apply Kruskal's algorithm to the following graph to obtain a minimum spanning tree:



4. The following is called Prim's algorithm:

   - *Initialize a tree with a single vertex, chosen arbitrarily from the graph.*
   - *Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and add it to the tree.*
   - *Repeat step 2 (until all vertices are in the tree).*

   Prove that, given a graph G as input, the output of Prim's algorithm is a minimum spanning tree of G.

5. Apply Prim's algorithm to the graph from exercise 2.

6. Use Prüfer codes, determine the number of trees on n vertices with a given degree sequence $d_1, d_2, \ldots, d_n$.

7. Consider an n-vertex complete graph with a selected edge e. Prove that the graph contains $2n^{n-3}$ spanning trees containing the edge e.

8*. We are given 99 cards, each of them containing a number from 1 to 99 (the numbers are not necessarily distinct). We know that whatever way we select some cards (at least one), the sum of the numbers on them is not divisible by 100. Show that the same number is written on all the cards.