Packing and covering - problem set 5

March 19, 2014.

1. Prove that for every convex disc C and for any point O in the interior of C, there exist points x, y on the boundary of C such that O is the midpoint of (x, y).

2. Show that every convex set C has a circumscribed square, that is, a square all of whose sides are tangent to C.

3. Remember that a function $f : \mathbb{R} \to \mathbb{R}$ is called *convex*, if for all $x, y \in \mathbb{R}$ and for all $t \in [0, 1]$, the following inequality holds:

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

   Prove Jensen's inequality by induction, that is, prove the following: Let ϕ be a convex real function and $x_1, ..., x_n$ be real values. For arbitrary non-negative real numbers $\lambda_1, \lambda_2, ..., \lambda_n$ satisfying $\sum \lambda_i = 1$, the following inequality holds:

$$\phi\left(\sum \lambda_i x_i\right) \leq \sum \lambda_i \phi(x_i).$$

4. For a planar graph G, we call the connected components of $\mathbb{R}^2 \setminus G$ the *faces* of G. Show that if G is a planar, connected graph with V vertices, E edges, and F faces, then $V - E + F = 2$.

5. Find a lattice packing of the plane with equilateral triangles that has density $2/3$.

6. Find a lattice covering of the plane with equilateral triangles that has density $3/2$.

7*. There are 100 smurfs standing in a row, with a red or blue hat on each of their heads. The evil Gargamel wants to feed them to his cat Azrael. There is only one way to escape: starting from the end of the line, each of the smurfs has to guess if his hat is red or blue. If at most one of them makes a mistake, then they are saved, otherwise, alas, Azrael eats them. The smurfs can set up a strategy beforehand, but cannot talk to each other afterwards; they can see only the smurfs standing ahead of them. Can you advise the smurfs a strategy that saves them from Gargamel?