Discrete mathematics - problem set 5

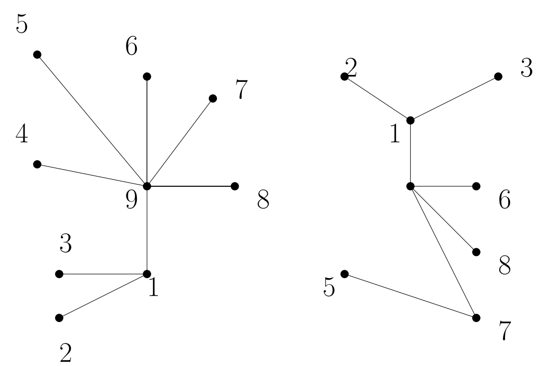
October 16, 2014.

1. Let G be a connected graph such that every spanning tree contains a given edge e. Show that e is a bridge, that is, if we remove e, the graph G becomes disconnected (Such an edge is often called a *bridge*.).

2. Let T be a tree. Prove that any new edge to T creates a unique cycle.

3. Prove that a graph G is a tree if and only if it is connected, and removing any edge makes is disconnected.

4. Recall that in Prüfer's proof of Cayley's theorem, every tree on a set of labeled vertices is encoded as a sequence of numbers. Compute the code of the following trees



5. Find the trees that have the following Prüfer sequences:
$$(4, 4, 3, 1, 1);\ \ (4, 2, 1, 1, 3).$$

6. Determine the trees whose Prüfer sequences are constant!

7. Show that the encoding in Prüfer's proof is surjective, that is, every sequence of $n-2$ numbers with entries from 1 to n is the code of a tree with vertices labeled $1, \ldots, n$.

8. Consider an n-vertex complete graph with a selected edge e. Prove that the graph contains $2n^{n-3}$ spanning trees containing the edge e.

9*. Jane wants to adopt three pets: a dog, a cat and a mouse. One day, she is going to an animal shelter, where there are 20 dogs, 20 cats and 20 mice. Each animal at the shelter likes exactly 10 animals of the two other species in total (for instance each cat likes in total exactly 10 dogs and mice from the shelter) and dislikes all the others. Assuming that the feelings between animals are mutual, can Jane (who was informed which animal likes who) always find 3 pets, so that they all like each other? Justify your answer!