Packing and covering - problem set 4

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1. Show that any convex disc permits an inscribed affine regular hexagon.

2. Prove that for any convex disc C, the difference region $D(C) = C + (-C)$ is convex and centrally symmetric about the origin. Furthermore, if p is a boundary point of $D(C)$, then there exist $q, q' \in \partial C$ such that $p = q - q'$ (where ∂C denotes the boundary of C).

The remaining exercises are related to the density of a collection of discs, that we define as follow. Let $\mathcal{C} = \{C_1, C_2, ...,\}$ be a collection of congruent copies of a convex disc C in the plane and let D be a domain. $\mathcal{C}$ is called a *covering* of D if $\cup_i C_i \supseteq D$. On the other hand, if $\cup_i C_i \subseteq D$ and no two distinct copies have a common interior point, then $\mathcal{C}$ is said to form a *packing* in D. If D is a bounded domain, then the *density* of the collection $\mathcal{C}$ with respect to D is defined as

$$d(\mathcal{C}, D) = \frac{\sum A(C_i)}{A(D)}$$

where the sum is taken over all i for which $C_i \cap D \neq \emptyset$. If D is the whole plane, then we define the upper and lower densities, denoted by $\overline{d}$ and $\underline{d}$, respectively, as follows. Let $Q(r)$ denote the square $[-r, r] \times [-r, r] = [-r, r]^2$, and introduce

$$\overline{d}(\mathcal{C}, \mathbb{R}^2) = \limsup_{r \to \infty} d(\mathcal{C}, Q(r))$$

$$\underline{d}(\mathcal{C}, \mathbb{R}^2) = \liminf_{r \to \infty} d(\mathcal{C}, Q(r))$$

If these two numbers coincide, then their common value is called the *density* of the collection $\mathcal{C}$ in the plane, and is denoted by $d(\mathcal{C}, \mathbb{R}^2)$.

Given a convex disc C in the plane, let

$$\delta(C) = \sup_{\mathcal{C}\text{packing}} \overline{d}(\mathcal{C}, \mathbb{R}^2),$$

where the supremum is taken over all packings in the plane with congruent copies of C.

3. Prove that if $\mathcal{C} = \{C_1, C_2, ...,\}$ is a collection of convex discs in the plane, then $\underline{d}(\mathcal{C}, \mathbb{R}^2)$ and $\overline{d}(\mathcal{C}, \mathbb{R}^2)$ are independent of the choice of origin.

4. Let $\mathcal{C}$ be a packing of congruent copies of a convex disc C and let us consider the following values

$$d(\mathcal{C}, D) = \frac{\sum_{C_i \cap D \neq \emptyset} A(C_i)}{A(D)}, \quad \mu(\mathcal{C}, D) = \frac{\sum_{C_i \subset D} A(C_i)}{A(D)}.$$

Prove that as $r \to \infty$,

$$\frac{d(\mathcal{C}, Q(r))}{\mu(\mathcal{C}, Q(r))} \to 1.$$

5. Let C be a convex disc in the plane. As above, we introduce the packing density of C as

$$\delta(C) = \sup_{\mathcal{C}\ packing} \overline{d}(\mathcal{C}, \mathbb{R}^2) = \sup_{\mathcal{C}\ packing} \limsup_{r\to\infty} d(\mathcal{C}, Q(r)),$$

where $\mathcal{C}$ is always a packing of congruent copies of C in the plane. Prove that this supremum is in fact a maximum, that is, there exists a packing $\mathcal{C}$ with $\delta(C) = d(\mathcal{C}, \mathbb{R}^2)$ as follows:

a) Show that we can choose an increasing sequence of real numbers $r_k \to \infty$ and corresponding packings $\mathcal{C}_k$ of congruent copies of C in $Q(r_k) = [-r_k, r_k]^2$, so that $r_k > k \cdot r_{k-1}$, and $d(\mathcal{C}_k, Q(r_k)) \to \delta(C)$ as $k \to \infty$.

b) Construct the packing $\mathcal{C}$ as follows: tile the annulus $[-r_k, r_k]^2 \backslash [-r_{k-1}, r_{k-1}]^2$ with disjoint copies of the square $[r_{k-2}, r_{k-2}]^2$ as dense as possible (there may remain gaps!); in each of these squares, place a translate of the packing $\mathcal{C}_{k-2}$. Prove that the density of this packing is $\delta(C)$.

6. Find a lattice such that if we place circles of radius one around each point of the lattice, then no two circles overlap and $\frac{\pi}{\sqrt{12}}$ of the plane is covered.

7*. Amélie and Alain play the following game. They place 1–franc coins on a circular table one by one; the coins can not overlap, so the next player must always put the coin on an empty spot. If the player cannot place a coin on the table, he/she loses. Assume that both of them have enough coins to cover the whole table. Alain is a gentleman and he lets Amélie start. Can she make sure she wins the game?