Packing and covering - problem set 3

March 4, 2014.

1. Let P be a convex polygon contained in a convex polygon Q. Show that $\mathrm{Per}(P) \leq \mathrm{Per}(Q)$.

2. Let C be a convex set and p_n be an n-gon with largest area inscribed in C. Show that $\dfrac{A(C) - A(p_n)}{A(C)} \leq \frac{c}{n^2}$ for some fixed constant c.

3. Euler's formula states that if a finite, connected, planar graph is drawn in the plane without any edge intersections, and v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely large region), then $v - e + f = 2$. Show that if G is a planar graph with n vertices, then it has at most $3n - 6$ edges and that, evenmore, if G is bipartite then it has at most $2n - 4$ edges.

4. A convex hexagon which is the image of a regular hexagon under an affine transformation is said to be *affinely regular*. Show that a convex hexagon $p_1, ..., p_6$ is affinely regular if and only if it is centrally symmetric and $\overrightarrow{p_2p_1} + \overrightarrow{p_2p_3} = \overrightarrow{p_3p_4}$.

5*. Suppose you have three boxes. The first one contains two one-dollar coins, the second one one-dollar coin and one-franc coin, and the third two one-franc coins. One of these boxes is chosen at random, and from it one of the coins is taken out at random. This coin turns out to be a one-dollar coin. What is the probability that the remaining coin, in the chosen box, is also a one-dollar coin?

6. (Optional exercise) The Gamma function is defined by $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$ It satisfies the following

$$\Gamma(1) = 1, \quad \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}, \quad \Gamma(2) = 1, \quad \Gamma(x+1) = x\Gamma(x), \quad \forall x > 0.$$

Using integration by parts, prove that

$$n \cdot \int_0^{\pi/2} cos^n \alpha \, d\alpha = (n-1) \int_0^{\pi/2} cos^{n-2} \alpha \, d\alpha.$$

By induction on n, show that

$$\int_0^{\pi/2} cos^n \alpha \, d\alpha = \frac{\sqrt{\pi} \cdot \Gamma(\frac{n+1}{2})}{2\Gamma(\frac{n}{2} + 1)}.$$

Let κ_d denote the volume of the unit ball in $\mathbb{R}^d$. Show that

$$\kappa_d = \kappa_{d-1} \int_{-1}^{1} (\sqrt{1 - x^2})^{d-1} dx.$$

Using the above formulas, eventually prove by induction on d that

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}.$$