Packing and covering - problem set 2

February 26, 2014.

1. Let C be the unit circle, and let P_n denote an n-gon of minimum area circumscribed about C, where $n \geq 3$. Calculate the area of P_n.

2. Let C be a convex disc in the plane, and let p_n denote an n-gon of maximum area inscribed in C, $n \geq 3$. Similarly to the methods seen in class, prove that

$$A(p_n) \geq \frac{A(p_{n-1}) + A(p_{n+1})}{2}.$$

3. Let C be a convex disc in the plane containing 0 and let the boundary of C be parametrized as:

(a) $x(\phi) = r(\phi)\cos(\phi)$, $y(\phi) = r(\phi)\sin(\phi)$, where $r(\phi)$ is a non-negative periodic function with period 2π, and ϕ runs from 0 to 2π. Prove that

$$A(C) = \int_0^{2\pi} \frac{r(\phi)^2}{2}\, d\phi.$$

(b) Assume that C has diameter 2, and $[-1,1] \subset C$. Let the boundary be $x(\psi) = \cos\psi$, $y(\psi) = h(\psi)\sin\psi$, where $h(\psi) > 0$ is a periodic function of period 2π. Show that

$$A(C) = \int_0^{2\pi} h(\psi)\sin^2\psi\, d\psi.$$

4. For two points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ in the plane, we define their distance to be

$$d_1(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|;$$

this is called the l_1-metric, or taxicab (Manhattan) distance. Let C be a circle of radius one. Show that for any set of 10 points $p_1, ..., p_{10}$ in the interior of C, there exist two indices $i, j \in \{1, ..., 10\}$, such that $d_1(p_i, p_j) < 4/3$.

5. Let C be a convex disc in the plane, and let T be the triangle of largest area inscribed in C. Show that in a suitable congruent copy of $2T$ contains C.

6. Determine the smallest area of a triangle circumscribed about the unit square $[0,1]^2$. Show that the proportion between the areas of the smallest circumscribing triangle and the square is bigger than in the case of a circular disc and the smallest triangle circumscribed about it.

7*. Can you hang your diploma on the wall with a piece of string using two nails, so that if you remove any of the nails, the picture falls down?