

Discrete mathematics - problem set 11

December 4, 2014.

1. We say that a tournament G has property S_2 if for any two vertices v_1 and v_2 of G , there exists a vertex u of G , such that $\overrightarrow{uv_1}$ and $\overrightarrow{uv_2}$ are both edges of G . Construct an example of a tournament that has property S_2 .
2. Let $G = (V, E)$ be a graph on an even number of vertices. We apply the following algorithm. First, we partition the vertices into two non-empty sets. Then for each vertex we compare the number of edges that connect it to a vertex in its own set to the number of edges that connect it to a vertex in the other set. If the former is larger then we move the vertex to the other set. We continue moving vertices until there is no more that needs to be moved. Show that the algorithm terminates after finitely many steps. Show that at the end, the number of edges between the two sets is at least half the total number of edges of G .
3. Let $G = (V, E)$ be a graph on an even number of vertices. Randomly select half of its vertices V_1 , and consider the bipartition of V into $V_1, V_2 = V \setminus V_1$. Show that with non-zero probability, the number of edges between V_1 and V_2 is at least $|E|/2$.
4. A tournament is a directed graph which contains exactly one directed edge connecting any two distinct vertices u and v : either \overrightarrow{uv} or \overrightarrow{vu} . A Hamiltonian path in a tournament is a directed path that goes through every vertex exactly once. Prove that for every $n \in \mathbb{N}$, there is a tournament on n vertices with at least $n!/2^{n-1}$ Hamiltonian paths.
- 5 *. Marie and Jean are playing the following game: they alternately choose numbers from among $1, 2, \dots, 9$ without being able to choose the same number more than once. The first to obtain 3 numbers summing up to 15 wins. Assuming that Marie starts, can she always win (that is, does there exist a winning strategy for the first player)? Justify your answer.