

Discrete mathematics - problem set 10

November 26, 2015.

1. Answer the following questions:
 - if A and B are disjoint events (they cannot occur at the same time), are they independent?
 - is it true that Ω and $A \subset \Omega$ are independent?
 - if A and B are independent, are A and \bar{B} independent? What about \bar{A} and \bar{B} ?
2. Let $\Omega = [0, 1]$ and $A = [1/2, 3/4]$. Draw the characteristic random variable for the event A .
3. What is the expected number of heads in 100 tosses of a fair coin? What if the coin lands heads with probability $1/10$?
4. A random graph $G(n, p)$ is a probability space of all labeled graphs on n vertices $\{1, 2, \dots, n\}$, where for each pair $1 \leq i < j \leq n$, (i, j) is an edge of $G(n, p)$ with probability p , independently of any other edge (you can think of a sequence of independent coin tosses for each edge). Compute the following:
 - a. the expected number of edges in $G(n, p)$;
 - b. the expected degree of a vertex in $G(n, p)$;
 - c. the expected number of triangles (cycles of length 3) in $G(n, p)$;
 - d. the probability that the degree of a given vertex v is at most k .
5. Let G be a graph with m edges, and let k be a positive integer. Prove that there is a coloring of the vertices of G with k colors so that at most m/k edges of G connect two vertices with the same color.
6. Recall that a tournament G has property S_2 if for any two vertices v_1 and v_2 of G , there exists a vertex u of G , such that $\overrightarrow{uv_1}$ and $\overrightarrow{uv_2}$ are both edges of G . Construct an example of a tournament that has property S_2 .
7. Let $G = (V, E)$ be a graph on an even number of vertices. Randomly select half of its vertices V , and consider the bipartition of V into $V_1 \cup V_2 = V$. Show that with non-zero probability, the number of edges between V_1 and V_2 is at least $|E|/2$.
8. A complete graph is a graph with any two vertices connected by an edge. Prove that the edges of an complete graph on n vertices can be colored with two colors so that at most $\binom{n}{k} 2^{1-\binom{k}{2}}$ of its complete k -vertex subgraphs are monochromatic.
- 9*. An EPFL student is doing crosscountry skiing from Zermatt to Gornergrat. He leaves from Zermatt at 8am and arrives in Gornergrat at 11am (he might even sleep on the way). The next day he goes back from Gornergrat to Zermatt (he leaves at 8 am and arrives at 11am and follows exactly the same way). Is it true that there is a place on the way from Zermatt to Gornergrat where the student found himself at the same hour and minute both days? Justify your answer.