Packing and covering - problem set 10

April 30, 2014.

1. Let T_d be the regular d-dimensional simplex embedded into $\mathbb{R}^{d+1}$ with vertices $e_1, \ldots, e_{d+1}$, the standard basis vectors (remember, e_i has the form $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 coordinate is at the i-th position). Calculate the circumradius and the inradius of T_d. What is the d-dimensional volume of T_d? (Recall that the volume of a cone in $\mathbb{R}^d$ is Ah/d, where h is the height of the cone, and A is the $(d-1)$–dimensional volume of its base.)

2. Let T_d be as above, and define the points $u_0, \ldots, u_d$ by taking u_k to be the centerpoint of the regular k-dimensional simplex with vertices $e_1, \ldots, e_{k+1}$. Let T' be the (d-dimensional) simplex with vertices $u_0, \ldots, u_d$. What is the volume of T'? For any k and j satisfying $1 \leq k \leq j \leq d$, calculate the inner product
$$\langle u_k - u_0, u_j - u_o \rangle.$$

3. Let $B^d + \mathcal{C}$ be a *maximal* packing of unit spheres in $\mathbb{R}^d$: that is, a packing, where no more sphere can be placed so that it is disjoint from the others. Prove that $2B^d + \mathcal{C}$ then forms a covering of the space. Show that in this packing, the diameter of any Voronoi cell is at most 4; moreover, these cells can be inscribed in a ball of radius 2.

4. Given any centrally symmetric convex body $C \subseteq \mathbb{R}^d$, let $\delta_T(C)$ denote the maximal density of a packing $\mathbb{R}^d$ with translates of C, and let $\theta_T(C)$ denote the minimal density of a covering of $\mathbb{R}^d$ with translates of C. Using the idea of the previous exercise, show that
$$\delta_T(C) \geq \frac{\theta_T(C)}{2^d} \geq \frac{1}{2^d}.$$

5*****. We are given 99 cards, each of them containing a number from 1 to 99 (the numbers are not necessarily distinct). We know that whatever way we select some cards (at least one), the sum of the numbers on them is not divisible by 100. Show that the same number is written on all the cards.