GRAPH THEORY
PROF. A. KUPAVSKII

Spring semester 2014
8 of May 2014

Midterm

Last name :
First name :
Sciper :
Section :

Exercise :	1	2	3	4	5	6	Σ
Grade :							

- Write your answers in the space provided under each question.
- You may not use a calculator on this midterm.
- No additional materials are permitted.
- Even if you cannot solve a problem, write down your ideas.

Time : 9:15 to 11:15

Exercice 1 : (30 points: 3 for each part)

For each of following 10 statements decide whether they are true or false.

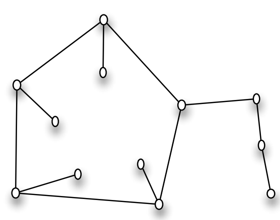
Justify your answer.

a. For every graph there is at most one spanning tree.

Solution. *False. Consider K_4.*

b. There is a graph with degree sequence $3,3,3,3,3,2,2,1,1,1,1,1$.

Solution. *True. This is the graph:*

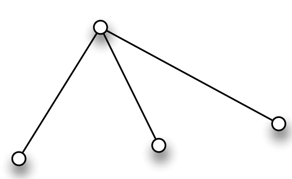


c. There is a tree with degree sequence $3,3,3,3,3,2,2,1,1,1,1,1$.

Solution. *False. Since it must have 11 edges.*

d. Every graph is the line graph of some other graph.

Solution. *False. The following graph is a counterexample.*



e .$K_{m,n}$ has an eulerian cycle if and only if m and n are even.

Solution. *True, a connected graph has an eulerian cycle iff every vertex has even degree.*

f. For every graph G with n vertices, $\chi(G) \leq n - \alpha(G) + 1$.

Solution. *True. Take the independent set with maximum size and color its vertices with a unique colour and color the remaining vertices of the graph with distinct colours.*

g. Every tree with 5 vertices has a perfect matching.

Solution. *False, becasue it must have an even number of vertices.*

h. For every graph G we have $k(G) < \delta(G)$ (recall that k denotes the (vertex) connectivity while δ denotes the minimum degree).

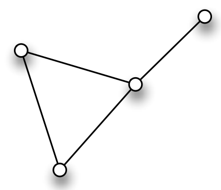
Solution. *False. We have always $k(G) \leq \delta(G)$ but the equality case can also be attained by considering for example K_n.*

i. There exists a graph G with $k(G/e) < k(G)$.

Solution. *True. Consider K_4.*

j.There exists a graph G with $k(G/e) > k(G)$.

Solution. *True. Take the one below.*



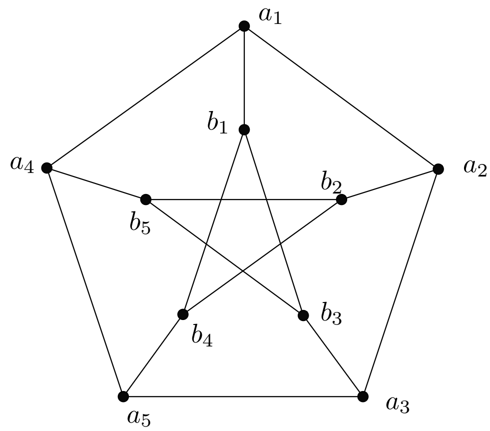
Exercice 2 : (20 points: 5 for each part.)

Let G be the Petersen graph (drawn in Figure 1).

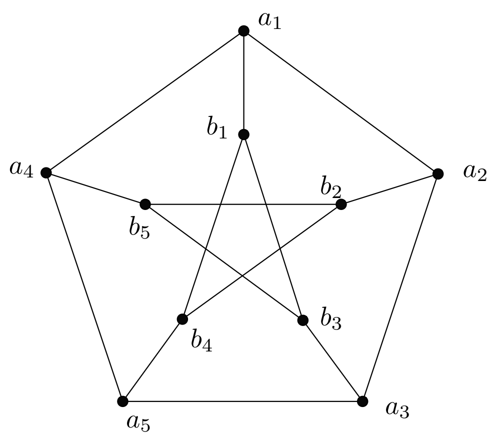
Use the drawings of G in Figure 1 to justify your answer.

a. Find a matching of maximum size in G.
b. Find $\alpha(G)$.
c. Find $\chi(G)$.
d. Find $\chi'(G)$.

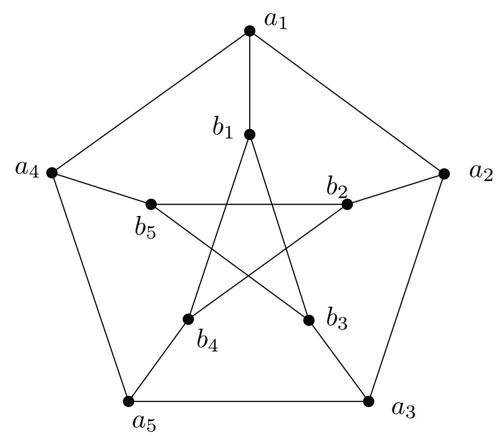
Solution. *a. It has a perfect matching: $\{a_1b_1, a_2b_2, a_3b_3, a_4b_5, a_4b_5\}$*
b. $\alpha(G) = 4$, consider the set $\{a_2, a_5, b_1, b_5\}$. If you choose any 5 vertices of the graph, by pigeonhole principle at least 3 of them would be from the a's (outer vertices) or b's (inner vertices), so there will be at least an adjacent pair between them.
c. $\chi(G) = 3$, consider this partition of the vertex set into three independent subsets: $\{a_1, a_5, b_2, b_3\}, \{a_2, a_4, b_1\}, \{a_3, b_4, b_5\}$ and since G has an odd cycle, its chromatic number can not be 2.
d. $\chi'(G) = 4$, consider the partition of its edge set into 4 independent subsets:
$\{a_1b_1, a_4b_5, a_2b_2, a_3a_5\}, \{a_1a_4, b_1b_3, a_2a_3, a_5b_4\}, \{a_1a_2, b_2b_5, b_1b_4, a_4a_5, a_3b_3\}, \{b_2b_4, b_3b_5\}$
Now by contrary assume that we can color its edges with just 3 colours red, blue and green. Then two of the colours must be used twice and the other one once in order to color the edges of cycle $a_1a_2a_3a_4a_5$. WLOG, we can assume that $\{a_1a_2, a_3a_5\}$ is red, $\{a_2a_3, a_1a_4\}$ is blue and $\{a_4a_5\}$ is green. Then we will get $\{a_2b_2, a_3b_3\}$ is green, $\{a_4b_5\}$ is blue and as a result $\{b_2b_5\}$ must be red. But then we can not color $\{b_3b_5\}$ and we will get a contradiction. The other cases can be checked similarly.



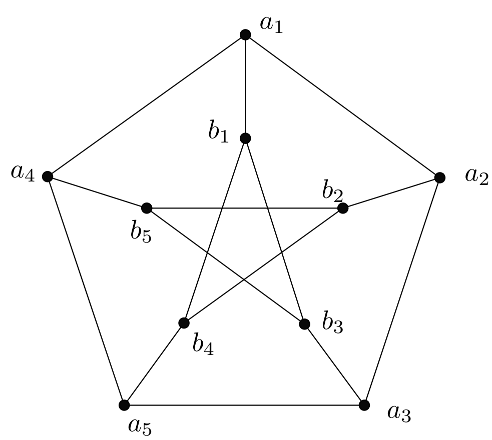
(a) Maximum matching



(b) Independent set



(c) Vertex colouring



(d) Edge colouring

Figure 1: The Petersen graph

Exercice 3 : (10 points.)

Let G be a graph with the property that any two odd cycles of G intersect. Show that $\chi(G) \leq 5$.

Solution. *Fix an odd cycle of G like C. Since every odd cycle in G has at least one vertex from G, deleting the vertices of C from G will result in a bipartite graph, because this graph would not have any odd cycle, so we can color it with 2 colours. Also C can be colored with 3 other colours, so in total G can be colored with 5 colours.*

Exercice 4 : (10 points.)

Show that every tree has at most one perfect matching.

Solution. *Consult exercise sheet 3, exercise 1.*

Exercice 5 : (15 points.)

If G is a 3-regular graph then the vertex connectivity of G is the same as the edge connectivity of G.

Solution. *Suppose k and k' are the vertex and edge connectivity numbers of G respectively. Then since $k \leq k'$, we are going to show that $k' \leq k$: By Menger theorem, we know that between every two vertices there are k' edge-disjoint paths. If there is a vertex common to two of these paths, then that vertex must have degree at least 4, which is impossible. So all of these k' paths are vertex independent and as a result we will get $k' \leq k$ as required.*

Exercice 6 : (15 points.)

Show that the petersen graph can be drawn in the plane in such a way that there are at most 2 pairs of edges that cross. As usual, we require that no edge goes over a vertex and that no edge crosses itself.

Solution.