Discrete mathematics - problem set 1

November 20, 2014.

Multiple choice

1. 2^{n+k}
2. True for some values of n and k.
3. Two. Using the fact that $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$, we obtain that $\sum_{k=1}^{n} k^2 = O(n^3)$ and $\sum_{k=1}^{n} k^2 = O(n^4)$ are correct.
4. One. The number of subsets of an n-element set is 2^n, we obtain that $f(n) = 2^n = O(5^n)$ is the only correct answer.
5. $\binom{11}{5}$.
6. There is no cycle containing p and q.

True-false questions

1. True.
2. True.
3. False.
4. True.
5. True.
6. True.
   $f(n) = n^{n-2}$, and therefore $n^{n-2} = O(n^n) = O((2^n)^n) = O(2^{n^2})$.
7. False.
   In Dilworth's theorem, the fact that the chain covering is minimal is crucial.

Problems

1. *Solution 1.*

   Define

   - A - the set of positive integers less than 462.

   - A_1 - the set of positive integers divisible by 3 and less than 462.

   - A_2 - the set of positive integers divisible by 7 and less than 462.

   - A_3 - the set of positive integers divisible by 11 and less than 462.

   The problem asks for the cardinality of the set

$$|A \backslash (A_1 \cup A_2 \cup A_3)| = |A| - |A_1 \cup A_2 \cup A_3| = 461 - |A_1 \cup A_2 \cup A_3|.$$

We use the principle of inclusion and exclusion for computing $|A_1 \cup A_2 \cup A_3|$. Therefore, we obtain

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$.

Note that $A_1 \cap A_2$ are all the positive integers less than 462 divisible by 21, $A_2 \cap A_3$ all positive integers less than 462 divisible by 77, $A_1 \cap A_3$ all positive integers less than 462 divisible by 33 and $A_1 \cap A_2 \cap A_3$ all positive integers less than 462 divisible by 231. We individually compute each of the values in the expression above, obtaining $|A_1| = 153$, $|A_2| = 65$, $|A_3| = 41$, $|A_1 \cap A_2| = 21$, $|A_1 \cap A_3| = 13$, $|A_2 \cap A_3| = 5$, $|A_1 \cap A_2 \cap A_3| = 1$. Therefore,

$$|A_1 \cup A_2 \cup A_3| = 461 - 153 - 65 - 41 + 21 + 13 + 5 - 1 = 240.$$

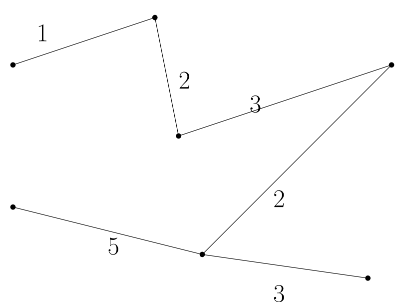
Solution 2. We can use Euler's function. Note that $\phi(462) = 120$ represents the number of positive integers relatively prime with 462 and less than 462, that is the positive integers less than 462, who are not divisible by neither of the numbers 2, 3, 7 or 11. So, $\phi(462)$ itself is **not** the final answer.

What we need is the result of the following expression

$$462 \cdot (1 - 1/3) \cdot (1 - 1/7) \cdot (1 - 1/11) = 240 = 2\phi(462).$$

2. We can choose either Kruskal's algorithm or Prim's algorithm (one is enough).

   A minimum spanning tree is as follows:



   Short description of Kruskal's algorithm:

   - Sort all the edges in non-decreasing order of their weight.

   - Pick the smallest edge. Check if it forms a cycle with the spanning tree formed so far. If cycle is not formed, include this edge. Else, discard it.

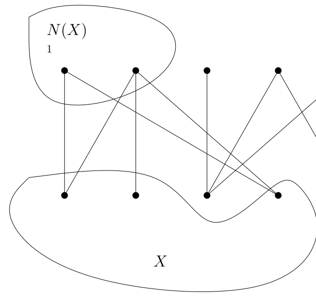
   - Repeat step until there are no edges left.

   Short description of Prim's algorithm:

   - Initialize a tree with a single vertex, chosen arbitrarily from the graph.

   - Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and add it to the tree.

   - Repeat step 2 (until all vertices are in the tree).



3. No, there is no perfect matching. Take for instance X as in the figure above. In this case, X has 3 elements, while $N(X)$ has only 2 elements. Therefore, by Hall's theorem, there is no perfect matching in the graph.

4. Note that, since S is contained in the square $[0, 2] \times [0, 2]$, its image under the function f must be contained in the following set of 9 elements

$$\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}.$$

   Since S contains 10 elements, by the pigeonhole principle, there must exist two elements (x_1, y_1) and (x_2, y_2), who have the same image under f, that is $f(x_1, y_1) = f(x_2, y_2)$, which completes the proof.