Lecture Notes
Geometric Graph Theory

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Contents

Chapter 1

Drawing Planar Graphs

A **graph** G consists of a finite set $V(G)$ of **vertices** (points) and a set $E(G)$ of **edges**, where every edge is a 2–element subset $\{u, v\} \subseteq V(G)$, $u \neq v$. For the sake of simplicity, an edge $\{u, v\}$ is often denoted by uv (or vu). Vertices u and v are called the **endpoints** of $uv \in E(G)$. If $uv \in E(G)$, then we say that u and v are **connected** (**joined** by an edge) in G, or that they are **adjacent**. A graph H is a **subgraph** of G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given a k–element set $V = \{v_1, v_2, \ldots, v_k\}$, the graphs P^{k-1} and C^k defined by

$$V(P^{k-1}) = V,\ E(P^{k-1}) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}\,;$$
$$V(C^k) = V,\ E(C^k) = \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\}$$

are called a **path** of length $k-1$ and a **cycle** of length k respectively. Obviously $P^{k-1} \subseteq C^k$.

The most natural way of representing a graph in the plane is to assign distinct points to its vertices and connect two points by a Jordan arc if and only if the corresponding vertices are adjacent. (A **Jordan arc** connecting two points $u, v \in \mathbb{R}^2$ is a continuous non-self-intersecting curve $\phi\colon\ [0, 1] \to \mathbb{R}^2$ with $\phi(0) = u$, $\phi(1) = v$.) The actual positions of the points and the arcs play no role in this representation. However, some representations (drawings) of a graph are much simpler than some others, and usually we also want to produce a visually pleasing diagram. For instance we may require our arcs to be straight-line segments, we may wish to avoid crossing arcs, etc.

Definition 1.1. A graph G that can be represented in the plane so that no two arcs meet at a point different from their endpoints is said to be **embeddable in the plane** or **planar**. A particular representation of a planar graph satisfying this property is called a **plane graph**.

It is easy to see that the graphs K_5 and $K_{3,3}$ depicted in Figure 1.1 are not planar. Indeed, if e.g. $K_{3,3}$ could be embedded in the plane, then the

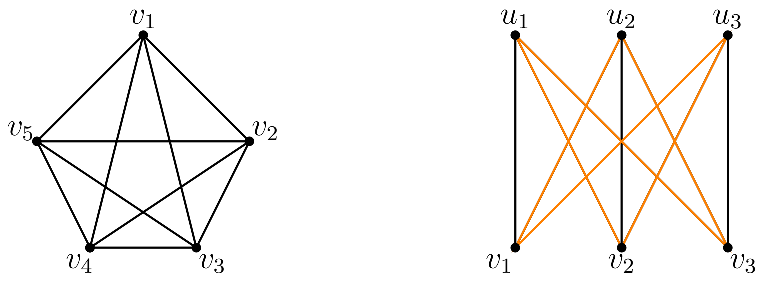


Figure 1.1: *Non-planar graphs: K_5 (left), $K_{3,3}$(right)*

arcs u_1v_2, v_2u_3, u_3v_1, v_1u_2, u_2v_3, v_3u_1 would form a closer Jordan curve φ, and every arc u_iv_i $(i = 1, 2, 3)$ would be entirely in the interior of φ or in the exterior of φ. Assume without loss of generality that u_1v_1 and u_2v_2 lie in the interior of φ. Then they should cross each other, contradicting our assumption. (A **closed Jordan** curve is a continuous non-self-intersecting curve φ: $[0, 1] \to \mathbb{R}^2$ with $\varphi(0) = \varphi(1)$. The complement of $\varphi([0, 1])$ falls into two disjoint connected open sets, whose closures are called the interior and exterior of φ.) Similarly, one can check that K_5 is not embeddable in the plane. In fact, a well-known **theorem of Kuratowski** states that a graph is not planar if and only if it has a subgraph that can be obtained from K_5 or $K_{3,3}$ by replacing the edges with paths, all of whose interior vertices are distinct.

Deleting any edge (say v_1v_5) from K_5, we obtain a planar graph. Moreover, this new graph can be embedded in the plane by using only straight-line segments (Figure 1.2). Does every planar graph have such a representation?

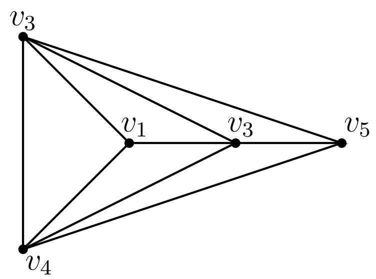


Figure 1.2: *A straight-line embedding of $K_5 - v_1v_5$*

As we shall see, the answer to this question is in the affirmative. Moreover, we will be able to impose some further restrictions on our drawings to ensure that the resulting diagrams are relatively balanced. But first we need some preparations.

1.1 Euler's formula

Let G be a graph. The **degree** $d_G(v)$ (or simply $d(v)$) of a vertex $v \in V(G)$ is the number of vertices adjacent to v (recall that v is never adjacent to itself!). Denoting the number of vertices and edges of G by $v(G)$ and $e(G)$, respectively, we clearly have

$$\sum_{v \in V(G)} d(v) = 2e(G). \tag{1.1}$$

Consequently, G must have a vertex whose degree is at most $2e(G)/v(G)$.

A graph G is said to be **connected**, if for any two vertices $v, v' \in V(G)$ there is a sequence of vertices $v_1 = v, v_2, v_3, \ldots, v_k = v'$ such that $v_i v_{i+1} \in E(G)$ for all i $(1 \leq i \leq k)$. In other words, G is connected if for any $v, v' \in V(G)$ there is a path $P_k \subseteq G$ with $v, v' \in V(P_k)$. If G is connected, then $d(v) \geq 1$ for any $v \in V(G)$, i.e., G cannot have any isolated vertices. It is easy to see that any connected graph has at least $v(G) - 1$ edges, and equality holds if and only if G has no cycle as a subgraph.

The arcs of a plane graph partition the rest of the plane into a number of connected components, called **faces**. Exactly one of these faces is unbounded, which is called the **exterior face**. The number of faces of a plane graph G is denoted by $f(G)$.

Theorem 1.1 (Euler's formula)**.** *If G is a connected plane graph, then*

$$v(G) - e(G) + f(G) = 2.$$

Proof. By induction on f. If $f(G) = 1$, then G has no cycle, thus by the above remark $e(G) = v(G) - 1$ and the assertion is true. Assume that $f(G) = f \geq 2$, and we have already proved the theorem for all connected plane graphs having fewer than f faces. Obviously, G must contain a cycle. Delete any edge e that belongs to a cycle of G. For the resulting plane graph $G - e$, $f(G - e) = f(G) - 1$, so we can apply the induction hypothesis to obtain

$$v(G) - (e(G) - 1) + (f(G) - 1) = 2.$$

□

Let G be a plane graph. If an edge (arc) e of G belongs to the boundary of only one face of G, then e is called a **bridge**. Let $F(G)$ denote the set of faces of G. For any $f \in F(G)$, let $s(f)$ be the number of **sides** of f, i.e., the number of edges belonging to the boundary of f, where all bridges are counted twice. Obviously,

$$\sum_{f \in F(G)} s(f) = 2e(G). \tag{1.2}$$

Corollary 1.2. *Let G be any plane graph with at least 3 vertices. Then*

(i) $e(G) \leq 3v(G) - 6$,

(ii) $f(G) \leq 2v(G) - 4$.

In both cases equality holds if and only if all faces of G have 3 sides.

Proof. It is sufficient to prove the statement for connected plane graphs. Clearly, $s(f) \geq 3$ for any face $f \in F(G)$. Then

$$3f(G) \leq \sum_{f \in F(G)} s(f) = 2e(G).$$

By Euler's formula, we obtain

$$v(G) - e(G) + \frac{2}{3}e(G) \geq 2,$$
$$v(G) - \frac{3}{2}f(G) + f(G) \geq 2,$$

as required. □

If $s(f) = 3$ for some face $f \in F(G)$, then f is called a **triangle**. If all faces of G are triangles, then G is a **triangulation**. It is easy to show that any plane graph can be extended to a triangulation by the addition of edges (without introducing new vertices). Corollary 1.2 implies that, if G is a triangulation, then it is **maximal** in the sense that no further edges can be added to G without violating its planarity.

The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colours necessary to colour the vertices of G so that no two vertices of the same colour are adjacent. According to the **four-colour theorem** of Appel and Haken, which settled a famous conjecture of Guthrie posed in the last century, the chromatic number of any planar graph is at most 4. (The graph in Figure 1.2 has chromatic number 4, showing that this bound cannot be improved.) The proof of Appel and Haken is quite complicated, and uses lengthy calculations by computers. However, a weaker statement can easily be deduced from Corollary 1.2.

Corollary 1.3. *If G is a planar graph, then $\chi(G) \leq 5$.*

Proof. By induction on $v(G)$. If $v(G) \leq 5$, then the statement is true, because one can assign a different colour to each vertex of G. Assume that $v(G) = v \geq 6$, and that we have already established the result for all planar graphs with fewer than v vertices.

It follows from Equation (1.1) and Corollary 1.2(i) that G has a vertex u with $d(u) \leq 5$. If $d(u) \leq 4$, then we apply the induction hypothesis to the graph $G - u$ obtained from G by the removal of u (and all edges incident to u). We get that the vertices of $G - u$ can be coloured by 5 colours so that no two vertices of the same colour are adjacent. Clearly, we can assign a colour to u, different from the (at most 4) colours used for its neighbours.

Suppose next that u is adjacent to 5 vertices u_i $(1 \leq i \leq 5)$. Since G is planar, it cannot contain K_5 as a subgraph. Thus, we can assume that, say u_1 and u_2 are not adjacent. Let G' denote the graph obtained from $G - u$ by merging u_1 and u_2. That is, $V(G') = (V(G) \setminus \{u, u_1, u_2\}) \cup \{u'\}$ and $E(G')$ consists of all edges of G, whose both endpoints belong to $V(G) \setminus \{u, u_1, u_2\}$, plus those pairs wu', for which $w \in V(G) \setminus \{u, u_1, u_2\}$ and either wu_1 or wu_2 belongs to $E(G)$. It is easy to see that G' is a planar graph, hence we can apply the induction hypothesis to obtain a proper colouring of the vertices of G' by 5 colours. If we assign the colour of u' to both u_1 and u_2, then we obtain a proper colouring of $G - u$ such that the vertices u_i $(1 \leq i \leq 5)$ have at most 4 different colours. Therefore, we can again colour u differently from its neighbours. □

1.2 Straight-line drawing

In this section we are going to show that every planar graph G can be embedded in the plane so that the arcs representing the edges of G are straight-line segments that can meet only at their endpoints. An embedding with this property is called a **straight-line embedding** of G. The existence of such an embedding was discovered independently by Fáry, Tutte and Wagner, but it also follows from an ancient theorem of Steinitz (see Exercise 1.7).

The proof presented here is based on a simple canonical way of constructing a plane graph, that will allow us to use an inductive argument for finding a proper position of the vertices one by one.

We need the following observation.

Lemma 1.4. *Let G be a plane graph, whose exterior face is bounded by a cycle $u_1, u_2, \ldots, u_k$. Then there is a vertex u_i $(i \neq 1, k)$ not adjacent to any u_j other than u_{i-1} and u_{i+1}.*

Proof. If there are no two non-consecutive vertices along the boundary of the exterior face that are adjacent, then there is nothing to prove. Otherwise, pick an edge $u_iu_j \in E(G)$, for which $j > i+1$ and $j-i$ is minimal. Then u_{i+1} cannot be adjacent to any element of $\{u_1, \ldots, u_{i-1}, u_{j+1}, \ldots, u_k\}$ by planarity, nor can it be adjacent to any other vertex of the exterior face different from u_i and u_{i+2}, by minimality of $j-i$. □

Let G be a graph (or a plane graph), $U \subseteq V(G)$. The subgraph of G **induced** by U is a graph (plane graph), whose vertex set is U and whose edge set consists of all edges of $E(G)$, both of whose endpoints belong to U.

Now we are in the position to establish the following useful theorem.

Theorem 1.5 (Canonical Construction of Triangulations). *Let G be a triangulation of n vertices, with exterior face uvw. Then there is a labelling of the vertices $v_1 = u, v_2 = v, v_3, \ldots, v_n = w$ satisfying the following conditions for every k $(4 \le k \le n)$:*

(*i*) *the boundary of the exterior face of the subgraph G_{k-1} of G induced by $\{v_1, v_2, \ldots, v_{k-1}\}$ is a cycle C_{k-1} containing the edge uv;*

(*ii*) *v_k is in the exterior face of G_{k-1}, and its neighbours in $V(G_{k-1})$ are some (at least two) consecutive elements along the path obtained from C_{k-1} by removal of the edge uv. (See Figure 1.3)*

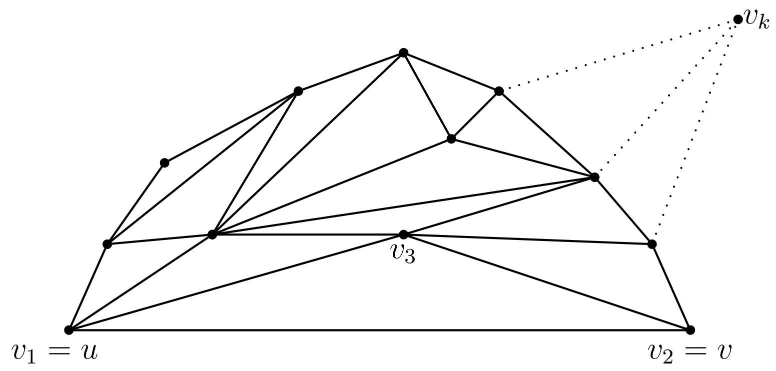


Figure 1.3: *G_{k-1} and v_k in the exterior*

Proof. The vertices $v_n, v_{n-1}, \ldots, v_3$ will be defined by reverse induction. Set $v_n = w$, and let G_{n-1} be the graph obtained from G by the deletion of v_n. Since G is a triangulation, the neighbours of w form a cycle C_{n-1} containing uv, and this cycle is the boundary of the exterior face of G_{n-1}.

Let $4 \le k \le n$ be fixed and assume that $v_n, v_{n-1}, \ldots, v_k$ have already been determined so that the subgraph G_{k-1} induced by $V(G) \setminus \{v_k, v_{k+1}, \ldots, v_n\}$ satisfies condition (i) and (ii). Let C_{k-1} denote the boundary of the exterior face of G_{k-1}. Applying Lemma 1.4 to G_{k-1}, we obtain that there is a vertex u' on C_{k-1}, different from u and v, which is adjacent only to two other points of C_{k-1} (i.e., to its immediate neighbours). Letting $v_{k-1} = u'$, the subgraph $G_{k-2} \subseteq G$ induced by $V(G) \setminus \{v_{k-1}, v_k, \ldots, v_n\}$ obviously meets the requirements. □

Using this theorem, we can easily prove the main result of this section.

Corollary 1.6. *Every planar graph has a straight-line embedding in the plane.*

Proof. It is sufficient to show that the statement is true for any **maximal** planar graph, i.e. for any graph that can be represented by a triangulation (see Exercise 1.4 and the remark after Corollary 1.2).

Let G be any triangulation with the canonical labelling $v_1 = u$, $v_2 = v$, $v_3, \ldots, v_n = w$, described above. We will determine the positions $f(v_k) = (x(v_k), y(v_k))$ of the vertices by induction on k.

Set $f(v_1) = (0,0)$, $f(v_2) = (2,0)$, $f(v_3) = (1,1)$. Assume that $f(v_1)$, $f(v_2)$, $\ldots$, $f(v_{k-1})$ have already been defined for some $k \ge 4$ such that, connecting the images of the adjacent vertex pairs by segments, we obtain a straight-line embedding of G_{k-1}, whose exterior face is bounded by the segments corresponding to the edges of C_{k-1}. Suppose further that

$$\begin{aligned} &x(u_1) < x(u_2) < \ldots < x(u_m), \\ &y(u_i) > 0 \qquad \text{for } 1 < i < m, \end{aligned} \tag{1.3}$$

where $u_1 = u$, u_2, u_3, $\ldots$, $u_m = v$ denote the vertices of C_{k-1} listed in cyclic order. By condition (ii) of Theorem 1.5, v_k is connected to u_p, u_{p+1}, $\ldots$, u_q for some $1 \le p \le q \le m$. Let $x(v_k)$ be any number strictly between $x(u_p)$ and $x(u_q)$. If we choose $y(v_k) > 0$ to be sufficiently large and connect $f(v_k) = (x(v_k), y(v_k))$ to $f(u_p)$, $f(u_{p+1})$, $\ldots$,$f(u_q)$ by segments, then we obtain a straight-line embedding of G_k meeting all the requirements (including the auxiliary Hypothesis (1.3) for the vertices of C_k). □

Note that by the same method we can also establish the existence of straight-line embeddings with some special geometric properties. For example, we can require that the segments corresponding to the edges of C_{k-1} form a convex polygon for every $k \ge 4$. (See also Exercises 1.9 and 1.10).

Corollary 1.7. *Let G be a planar graph with n vertices and $3n-6$ edges. Then there are a labelling of the vertices $v_1, v_2, \ldots, v_n$ and a straight-line embedding of G such that*

(i) *the image of the subgraph of G induced by $\{v_1, v_2, \ldots, v_{k-1}\}$ is a triangulated convex polygon C_{k-1},*

(ii) *the image of v_k lies in the exterior of C_{k-1}, for every $4 \leq k \leq n$.*

The same technique can be used to obtain a different kind of representation of planar graphs, found by Rosenstiehl and Tarjan.

Corollary 1.8. *The vertices and the edges of any planar graph can be represented by horizontal and vertical segments, respectively, such that*

(i) *no two segments have an interior point in common,*

(ii) *two horizontal segments are connected by a vertical segment if and only if the corresponding vertices are adjacent.*

Proof. As in the proof of Corollary 1.6, it is sufficient to establish the statement for triangulations.

Let G be any triangulation with canonical labelling $v_1 = u$, $v_2 = v$, v_3, $\ldots$, v_n. To every v_k we shall assign a horizontal segment $s(v_k)$ whose endpoints are (x_k, k) and (x'_k, k). Set $x_1 = 0, x'_1 = 2, x_2 = 2, x'_2 = 4, x_3 = 1, x'_3 = 3$. Assume that $s(v_1), s(v_2), \ldots, s(v_{k-1})$ have already been determined for some $k \geq 4$ such that the segments corresponding to adjacent point pairs can be connected by vertical segments, i.e., the subgraph $G_{k-1} \subseteq G$ induced by $\{v_1, v_2, \ldots, v_{k-1}\}$ has a representation satisfying conditions (i) and (ii). Let $u_1 = u$, u_2, $\ldots$, $u_m = v$ denote the vertices of the exterior face of G_{k-1}, listed in cyclic order. Suppose further that the upper envelope of the segments $s(v_1), s(v_2), \ldots, s(v_{k-1})$ consists of some portion of $s(u_1), s(u_2), \ldots, s(u_m)$, in this order. (A point $b \in s(v_i)$ belongs to the **upper envelope** of $s(v_1), s(v_2), \ldots, s(v_{k-1})$, if the vertical ray starting from b and pointing upwards does not intersect any other segment $s(v_j)$, $1 \leq j \leq k-1$).

By condition (ii) of Theorem 1.5, v_k is connected to $u_p, u_{p+1}, \ldots, u_q$ for some $1 \leq p \leq q \leq m$. Let b and b' be any interior points of those portions of $s(u_p)$ and $s(u_q)$, respectively, that belong to the upper envelope of $s(v_1), s(v_2), \ldots, s(v_{k-1})$. Letting x_k and x'_k be equal to the x–coordinates of b and b', respectively, we obtain a representation of G_k with the required properties (see Figure 1.4). □

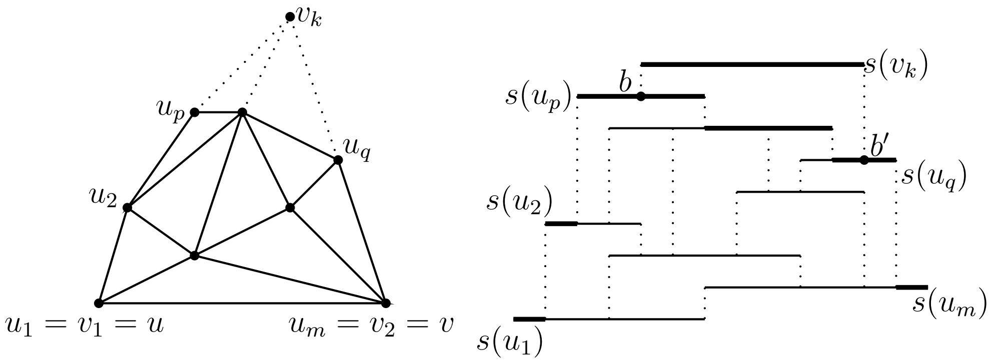


Figure 1.4: *Illustration to Corollary 1.8. The thick segments are the pieces of the upper envelope of* $s(v_1), s(v_2), \ldots, s(v_{k-1})$.

1.3 Drawing a planar graph on a grid

In the previous section we have seen that any planar graph has a straight-line embedding (Corollary 1.6). However, the solution has a serious drawback: as we embed the vertices recursively in the plane, we may be forced to map a new vertex far away from all previous points, so that the size of the picture may increase exponentially with the number of vertices. To put it differently, if we want to view the resulting drawing on a terminal screen, then many points will bunch together and become indistinguishable. To handle this problem, in this section we shall restrict our attention to straight-line drawings, where each point is mapped into a **grid point**, i.e., a point with integer coordinates. Our goal is to minimize the size of the grid needed for the embedding of any planar graph of n vertices. The set of all grid points (x, y) with $0 \leq x \leq m$, $0 \leq y \leq n$ is said to be an **m by n grid**.

Theorem 1.9. *Any planar graph with n vertices has a straight-line embedding on the $2n-4$ by $n-2$ grid.*

Proof. It suffices to prove the theorem for triangulations. Let G be a triangulation with exterior face uvw, and let $v_1 = u$, $v_2 = v$, v_3, $\ldots$, $v_n = w$ be a canonical labelling of the vertices (see Theorem 1.5).

We are going to show by induction on k that G_k, the subgraph of G induced by $\{v_1, v_2, \ldots, v_k\}$, can be straight-line embedded on the $2k-4$ by $k-2$ grid, for every $k \geq 3$. Let f_3 be the following embedding of G_3:

$$f_3(v_1) = (0,0), f_3(v_2) = (2,0), f_3(v_3) = (1,1).$$

Suppose now that for some $k \geq 4$ we have already found an embedding $f_{k-1}(v_i) = (x_{k-1}(v_i), y_{k-1}(v_i))$, $1 \leq i \leq k-1$, with the following properties:

(a) $f_{k-1}(v_1) = (0,0)$, $f_{k-1}(v_2) = (2k-6, 0)$;

(b) If $u_1 = u$, u_2, …, $u_m = w$ denote the vertices of the exterior face of G_{k-1} in cyclic order, then

$$x_{k-1}(u_1) < x_{k-1}(u_2) < \ldots < x_{k-1}(u_m);$$

(c) The segments $f_{k-1}(u_i)f_{k-1}(u_{i+1})$, $1 \le i < m$, all have slope $+1$ or -1.

Note that (c) implies that the Manhattan distance $|x_{k-1}(u_j) - x_{k-1}(u_i)| + |y_{k-1}(u_j) - y_{k-1}(u_i)|$ between the image of any two vertices u_i and u_j on the exterior face of G_{k-1} is even. Consequently, if we take a line with slope $+1$ through u_i and a line with slope -1 through u_j, then they always intersect at a grid point $P(u_i, u_j)$.

Let $u_p, u_{p+1}, \ldots, u_q$ be the neighbours of v_k in G_k $(1 \le p < q \le m)$. Clearly, $P(u_p, u_q)$ is a good candidate for $f_k(v_k)$, except that we may not be able to connect it to e.g. $f_{k-1}(u_p)$ by a segment avoiding $f_{k-1}(u_{p+1})$. To resolve this problem, we have to modify f_{k-1} before embedding v_k. We shall move the image of $u_{p+1}, u_{p+2}, \ldots, u_m$ one unit to the right, and then move the images of $u_q, u_{q+1}, \ldots, u_m$ to the right by an additional unit. That is, let

$$\tilde{x}_k(u_i) = \begin{cases} x_{k-1}(u_i), & \text{for } 1 \le i \le p, \\ x_{k-1}(u_i) + 1, & \text{for } p < i < q, \\ x_{k-1}(u_i) + 2, & \text{for } q \le i \le m, \end{cases}$$

$$y_k(u_i) = y_{k-1}(u_i), \quad \text{for } 1 \le i \le m\,,$$

and let $f_k(v_k)$ be the point of intersection of the lines of slope $+1$ and -1 through $f_k(u_p)$ and $f_k(u_q)$, respectively. Of course, $f_k(v_k)$ is a grid point that can be connected by disjoint segments to the points $f_k(u_i) = (x_k(u_i), y_k(u_i))$, $p \le i \le q$, without intersecting the polygon $f_k(u_1)f_k(u_2)\ldots f_k(u_m)$. However, as we move the image of some u_i, it may be necessary to move some other points (not on the exterior face) as well, otherwise we may create crossing edges.

In order to tell exactly which set of points has to move together with the image of a given exterior vertex u_i, we define recursively a total order '$\prec$' on $\{v_1, v_2, \ldots, v_n\}$. Originally, let $v_1 \prec v_3 \prec v_2$. If the order has already been defined on $\{v_1, v_2, \ldots, v_{k-1}\}$, then insert v_k just before u_{p+1}. According to this rule, obviously

$$u_1 \prec u_2 \prec \cdots \prec u_m\,.$$

Now we can extend the definition of f_k to the interior vertices of G_{k-1}, as follows. For any $1 \leq i \leq k-1$, let

$$\tilde{x}_k(v_i) = \begin{cases} x_{k-1}(v_i), & \text{if } v_i \prec u_{p+1}, \\ x_{k-1}(v_i)+1, & \text{if } u_{p+1} \preceq v_i \prec u_q, \\ x_{k-1}(v_i)+2, & \text{if } u_q \preceq v_i, \end{cases}$$

$$y_k(v_i) = y_{k-1}(v_i)\,.$$

Evidently, f_k satisfies conditions (a),(b) and (c).

To complete the proof, it remains to verify that f_k is a straight-line embedding, i.e., no two segments cross each other. A slightly stronger statement follows by straightforward induction.

Claim 1.10. *Let $f_{k-1} = (x_{k-1}, y_{k-1})$ be the straight-line embedding of $G_k - 1$, defined above, and let $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$. For any $1 \leq i \leq k-1$, $1 \leq j \leq m$, let*

$$x(v_i) = x_{k-1}(v_i) + \alpha_1 + \alpha_2 + \cdots + \alpha_j \text{ if } u_j \preceq v_i \prec u_{j+1}\,,$$

$$y(v_i) = y_{k-1}(v_i)\,.$$

Then $f'_{k-1} = (x, y)$ is also a straight-line embedding of G_{k-1}.

The claim is trivial for $k = 4$. Assume that it has already been confirmed for some $k \geq 4$, and we want to prove the same statement for G_k. The vertices of the exterior face of G_k are $u_1, \ldots, u_p, v_k, u_q, \ldots, u_m$. Fix now any nonnegative numbers $\alpha(u_1), \ldots, \alpha(u_p), \alpha(v_k), \alpha(u_q), \ldots, \alpha(u_m)$. Applying the induction hypothesis to G_{k-1} with $\alpha_1 = \alpha(u_1), \ldots, \alpha_p = \alpha(u_p), \alpha_{p+1} = \alpha(v_k) + 1, \alpha_{p+2} = \cdots = \alpha_{k-1} = 0, \alpha_q = \alpha(u_q) + 1, \alpha_{q+1} = \alpha(u_{q+1}), \ldots, \alpha_m = \alpha(u_m)$, we obtain that the restriction of f'_k to G_{k-1} is a straight-line embedding. To see that the edges of G_k incident to v_k do not create any crossing, it is enough to notice that f_k and f'_k map $\{u_{p+1}, \ldots, u_{q-1}\}$ into congruent sets. $\square$

Chapter 2

Characterization of planar graphs

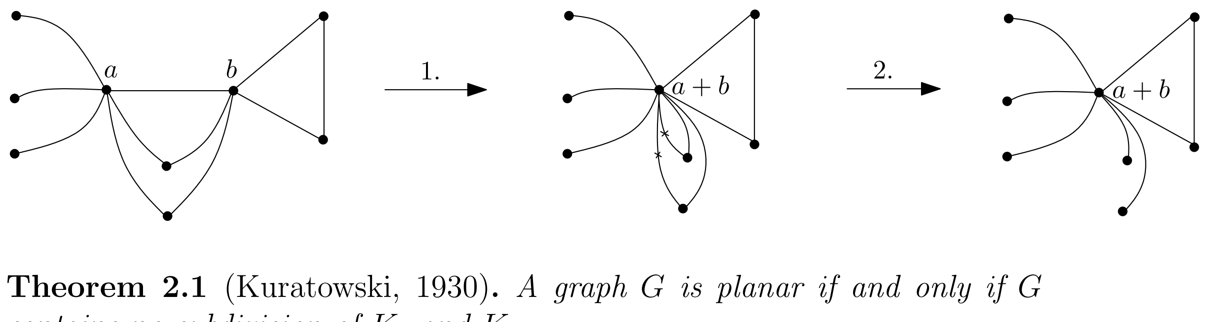
We begin this chapter with some criteria for graphs to be planar.

Definition 2.1. Take a graph G and put additional vertices arbitrarily on the edges of G (but not on their crossings). This divides the original edges of G into smaller ones. The resulting graph is called a **subdivision** of G.

Definition 2.2. A graph G contains H as a **minor** if H can be obtained from G by *shrinking* edges and deleting edges and vertices.

Shrink an edge ab means the following:

1. Identify a and b.
2. Remove all the parallel edges.



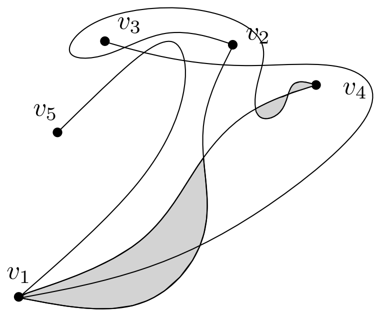
Theorem 2.1 (Kuratowski, 1930)**.** *A graph G is planar if and only if G contains no subdivision of K_5 and $K_{3,3}$.*

Theorem 2.2 (Wagner,1937)**.** *A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a minor.*

Theorem 2.3 (Hanani-Tutte 1934/1970)**.** *A graph G is planar $\iff$ G can be drawn so that any two edges cross an even number of times.*

Proof. (sketch) " $\Rightarrow$ " This direction is trivial: If G is planar then it can by definition be drawn without crossings, i.e. each pair of edges cross zero times and zero is an even number.
" $\Leftarrow$ " This direction can be proved in an easy way using **Kuratowski's Theorem**. Namely, we have to show that G has no subdivision of K_5 and $K_{3,3}$.

As an example we show that G does not contain K_5. Assume the contrary. Then there exists a drawing of K_5 such that any two edges cross an even number of times. Take a vertex v_1, and let the edges $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ leave v_1 in this clockwise order in a small neighborhood of v_1. Of course, outside this neighborhood these edges may cross one another. Consider the image of the triangle $v_1v_2v_4$. It is a closed, possibly self-intersecting curve γ. It divides the plane into several regions. Notice that these regions can be colored by two colors (white and red, say) such that no two regions whose boundaries share an arc get the same color. Notice that the initial portions of the edges v_1v_3 and v_1v_5 around v_1 belong to regions of opposite colors. Assume that the initial portion of v_1v_3 in a small neighborhood of v_1 runs in the red region.



According to our assumptions, the edge v_1v_5 must cross each of the edges v_1v_2, v_2v_4, v_4v_1 an even number of times. Therefore, the curve v_1v_5 crosses γ an even number of times, and after each crossing it switches colors. This yields that v_5 must lie in the white region. Since the edge v_3v_5 has to cross γ an even number of times, v_3 has to lie in a white region too. But since the initial portion of v_1v_3 lies in the red region and v_3 in the white, v_1v_3 has to cross γ an odd number of times. This is a contradiction to the assumption that any two edges cross an even number of times.

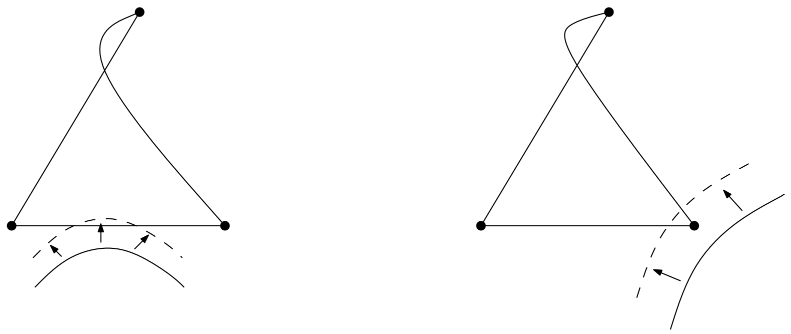
$\square$

There is a stronger version of the above theorem:

Theorem 2.4 (Hanani-Tutte, strong version)**.** *A graph G is planar $\Longleftrightarrow$ G can be drawn so that any two* ***non-adjacent*** *edges cross an even number of times.*

Proof. For the first direction, the same argument applies as before. For the second direction we again use **Kuratowski's Theorem** and show as an example that G does not contain a subgraph of type K_5.

Take a drawing of K_5. We are going to prove that the parity of the total number N of crossings for all nonadjacent pairs of edges does not change during any continuous transformation of a drawing of K_5. Since in the usual starlike straight-line drawing, the number N is odd (is five), it can never become even. To see this, take an edge $e = v_4v_5$ of K_5 and slightly deform it. We only have to check how the intersection between this edge and the edges of the triangle $T = v_1v_2v_3$ changes. As we pull e through a vertex of T, the total number of crossings between e and T changes by two. The possible two cases are considered in the following figure



If during the small deformation e is not pulled through any vertex, the total number of crossings between e and T can change only by two (or zero).

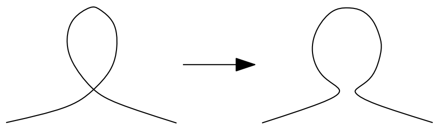
Similar arguments apply for $K_{3,3}$ in place of K_5.

□

There is another proof for the weaker version, i.e., for the first Hanani-Tutte theorem we stated:

Proof. (*of the weaker version of Hanani-Tutte-Theorem*)
We may assume that G is connected, since components may be redrawn arbitrarily far apart. Fix a plane drawing D of G in which every pair of edges intersects an even number of times. We prove the result by induction on the number of edges in G. To make the inductive step work, we keep track of the *rotation* of each vertex, that is, the cyclic order in which edges leave the vertex in the drawing. The mapping from the vertices of G to their rotations is called the *rotation system* of D. We will prove the following

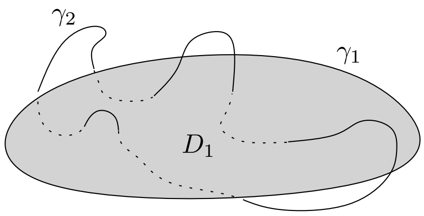
stronger statement:
If D is a drawing of a *multigraph* G so that any pair of edges intersects an even number of times in D, then G is planar and can be embedded without changing the rotation system. We begin with the inductive step: if there are at least two vertices in G, then there is an even edge $e = uv$. Pull v towards u until there remains no crossing between v and u.
Since e was an even edge, the edges incident to v remain even. The pull move will introduce self-intersections in curves that intersect e and are adjacent to v. Since drawings are typically defined not to have self-intersections, we remove them as shown on Figure.



Now that uv no longer has any intersections, we shrink it (but do not as usual remove resulting loops or parallel edges) to obtain a new graph G in which the rotations of u and v are combined appropriately. By the inductive assumption, there is a planar drawing of G respecting the rotation system. In such a drawing, we can simply split the vertex corresponding to u and v, reintroducing the edge e between them without any intersections. Hence G is planar respecting the rotations of all its vertices.
If G contains only a single vertex, then it might have several loops attached to it. Since all the loops in G are even, it cannot happen that we find edges leaving in order a, b, a, b since this would force an odd number of intersections between a and b. Hence, if we consider the regions enclosed within the two loops in a small enough neighborhood of the vertex, either they are disjoint or one region contains the other. Then it is easy to show that there must be a loop whose ends are consecutive in the rotation system. Removing this edge we obtain a smaller graph G which, by inductive assumption, can be drawn without intersections and with the same rotation system. We can then reinsert the missing loop at the right location in the rotation system by making it small enough. In the base case, we simply draw a single vertex with no edges. □

Our next subject are graphs in higher dimensions. A graph can be drawn in $\mathbb{R}^3$ in the following way: the vertices are points in $\mathbb{R}^3$ and the edges are simple curves such that they do not pass through any vertex and do not cross any other edge.

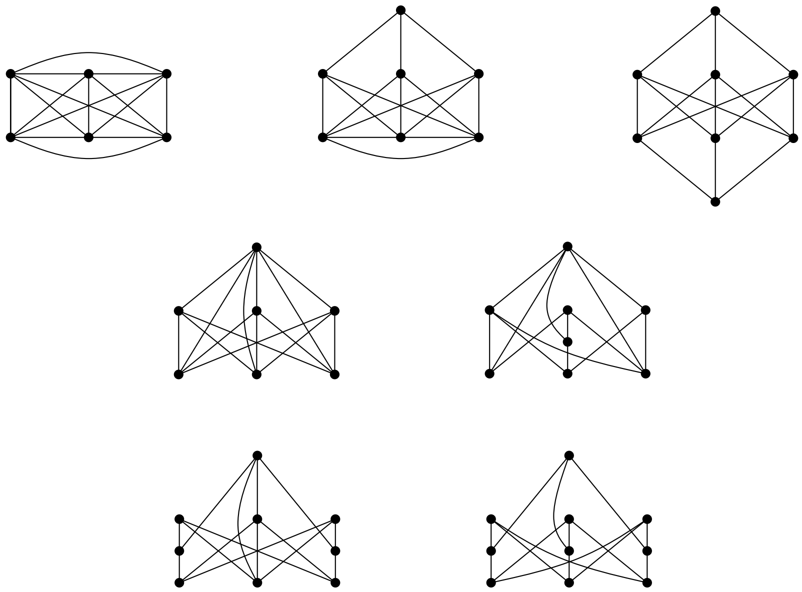
Definition 2.3. (i) Let γ_1, γ_2 be two closed curves. Denote a surface of which γ_1 is the boundary by D_1. Note that this can without loss of generality be seen as a disc. Then $\gamma_1,\ \gamma_2$ are called **linked** if the number of times γ_2 intersects D_1 from above is not equal to the number of times it intersects D_1 from below.



(ii) Two cycles $C_1,\ C_2$ are **linked** if the corresponding closed curves $\gamma_1,\ \gamma_2$ are linked.

(iii) G is a **linkless graph** if it can be drawn in $\mathbb{R}^3$ so that no two disjoint cycles are drawn as linked curves.

Theorem 2.5 (Robertson-Seymour-Thomas)**.** *A graph G has a linkless embedding in $\mathbb{R}^3$ if and only if G has no minor belonging to the Petersen family (shown on Figure).*



Example 2.1. *K_6 is not a linkless graph, i.e., it cannot be drawn without two linked cycles.*

One of the most important theorems about respresentation of planar graphs is the following Koebe representation theorem.

Theorem 2.6. *(Koebe 1936) The vertices $v \in V(G)$ of any planar graph G can be represented by disks D_v such that D_u and D_v are tangent to each other if and only if $uv \in E(G)$, otherwise $D_u \cap D_v = \emptyset$.*

Theorem 2.7. *(de Fraysseix, de Mendex, Rosenstiehl 1994) The vertices v of any planar graph G can be represented by non-overlapping triangles T_v so that T_u and T_v have a point of contact if and only if $uv \in E(G)$.*

These two theorems give rise to the following question:

Question. *Is it true that the vertices of every planar graph can be represented by (pseudo-)segments so that two of them intersect if and only if the corresponding vertices are adjacent?*

The following two theorems answer part of this problem.

Theorem 2.8. *(Hartman, Newman, Ziv; de Fraysseix, de Mendez, Pach 1991) True for bipartite planar graphs.*

Theorem 2.9. *(Castro, Cobos, Dana, Màrquez, Noy 1999) True for triangle-free planar graphs.*

Chapter 3

Crossings: few and many

3.1 Turan's Brick Factory Problem

In 1944, Turan posed the following problem. Suppose that there are m brick factories and n storage places. How can we connect every brick factory to every storage place with paths so that the number of crossing paths is minimum? This problem can be modeled as follows. What is the minimum number of crossing pairs of edges in a drawing of the graph $K_{m,n}$ on the plane?

For an arbitrary graph G we define $cr(G)$ as the minimum number of crossing pairs of edges over all possible drawings of G on the plane. So, the Brick factory problem of Turan is to find $cr(K_{m,n})$.

Conjecture 3.1 (Zarankiewicz). $cr(K_{n,m}) = [\frac{m}{2}][\frac{m-1}{2}][\frac{n}{2}][\frac{n-1}{2}]$.

Suppose that the vertices of $K_{m,n}$ are partitioned into two parts A and B with $|A| = m$ and $|B| = n$ and every vertex in A is connected to every vertex in B. Notice that there is a simple drawing of $K_{n,m}$ which achieves the bound given in the above conjecture. Namely, place the vertices of A in the following points of x-y plane:

$$(0, -[\frac{m}{2}]), (0, -[\frac{m}{2}] + 1), \ldots (0, -1), (0, 1), (0, 2), \ldots, (0, [\frac{m}{2}] - 1)$$

and the vertices of B on the following points:

$$(-[\frac{n}{2}], 0), (-[\frac{n}{2}] + 1, 0), \ldots (-1, 0), (1, 0), (2, 0), \ldots, ([\frac{n}{2}] - 1, 0)$$

and then join every A-vertex to every B-vertex with a straight line segment. So the problem is that whether this is an optimal drawing.

In the next theorem we will show $cr(K_{n,n})$ and $cr(K_n)$ are of order n^4.

Theorem 3.2. $\lim_{n\to\infty} \frac{cr(K_{n,n})}{\binom{n}{2}^2}$ *and* $\lim_{n\to\infty} \frac{cr(K_n)}{\binom{n}{4}}$ *exist and both are positive numbers.*

Proof. We will prove the theorem only for the graph K_n. Proof for the graph $K_{n,n}$ is similar. First of all notice that for every drawing of K_n and every four vertices of it, there are at most three possible crossings among the edges between these four vertices. This observation shows that $\frac{cr(K_n)}{\binom{n}{4}}$ never exceeds 3. Now, in order to show that the limit exists, it is sufficient to show that the sequence $\frac{cr(K_n)}{\binom{n}{4}}$ $n = 1, 2, 3, \ldots$ is an increasing sequence. Since every increasing upper bounded sequence whose first term is positive has a positive limit, Theorem 3.2 can be proved. To complete the proof, we need to show that for every integer n we have $\frac{cr(K_n)}{\binom{n}{4}} \leq \frac{cr(K_{n+1})}{\binom{n+1}{4}}$. If we expand the binomial coefficients in both sides and ignore the common factors in both sides, we can observe that this inequality is equivalent to the following inequality

$$\frac{cr(K_n)}{n} \geq \frac{cr(K_{n-1})}{n-4} \text{ or equivalently } (n-4)\,cr(K_n) \geq n\,cr(K_{n-1}).$$

Fix a drawing of K_n. There are at least $cr(K_n)$ crossings in this drawing. Now we present a lower bound on the number of the crossing of the graph K_n. Removing each vertex of K_n yields a copy of K_{n-1} which has at least $cr(K_{n-1})$ crossings. Therefore there must be at least $n \cdot cr(N_{n-1})$ crossings in K_n. But notice that every crossing in K_n is counted precisely $n-4$ times. Therefore, the number of the crossings in K_n should be at least $\frac{n \cdot cr(K_{n-1})}{n-4}$. This shows that $cr(K_n) \geq \frac{n \cdot cr(K_{n-1})}{n-4}$. $\square$

Observe that the above proof tells us more than just existence of a limit. What it says is that the sequence $\frac{cr(K_n)}{\binom{n}{4}}$ is an increasing sequence. Therefore, every term of this sequence is a lower bound for $\lim_{n\to\infty} \frac{cr(K_n)}{\binom{n}{4}}$.

3.2 Conway's Thrackle Conjecture

A *thrackle* is a graph drawn in the plane so that the edges are represented by Jordan curves, any pair of which either meet at a common vertex or cross precisely once.

Conjecture 3.3. *In every thrackle, the number of edges is less than or equal to the number of vertices.*

Conway's thrackle conjecture is analogous to the following theorem of Fisher, known as Fisher's inequality.

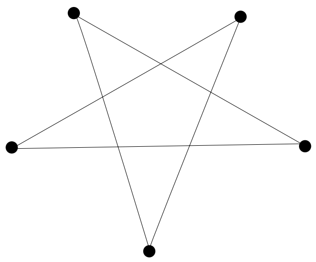


Figure 3.1: *Thrackle drawing of C_5*

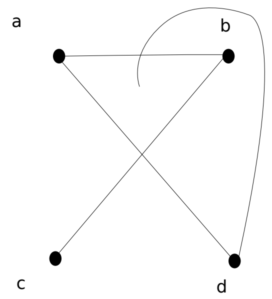


Figure 3.2: C_4

Theorem 3.4. *Given a system of subsets of a set X such that any pair have precisely one element in common, the number of the sets is less than or equal to the size of X.*

An example of a thrackle is the graph C_5. This can be easily seen from the star-like drawing of C_5 (Figure 3.1). But C_4 is not a thrackle. In fact if the vertices of C_4 are a, b, c, d and each vertex is joint to the next vertex in this order then in every thrackle drawing of C_4, there is only one possible configuration for the path $a-b-c-d$ shown in Figure 3.2. Now these three edges create a triangle whose one side is the edge bc. So, if we consider the edge da it must cross the edge bc. So it has to get into the triangle and when it goes out of the triangle it either crosses the edge bc for the second time or it must cross one of the other edges. None of these situations is allowed in a thrackle. This argument shows that C_4 is not a thrackle. Clearly, every subgraph of a thrackle is also a thrackle. This together with the previous observations shows that if G is a thrackle then G has no C_4 as a subgraph.

Theorem 3.5 (Erdős, Kővári-Sós-Turán)**.** *Any graph G with n vertices with no C_4 as a subgraph has at most $n^{\frac{3}{2}}$ edges.*

Proof. Suppose that G is a graph with n vertices with no C_4 as its subgraph. We count the number of paths of length 2 in G in two ways. Since G has no C_4, every pair of its vertices have at most one common neighbor and therefore the number of 2-paths in G is at most $\binom{n}{2}$. Now we count the number of 2-paths as follows. Let v be a vertex of G of degree d. Every pair of the neighbors of v form a path of length 2 and conversely every such path is obtained this way precisely once (just consider the middle point of the 2-path). So, the number of 2-paths in G is equal to:

$$\sum_{i=1}^{n} \binom{d_i}{2}$$

where d_i's are the degrees of the vertices of G. So we should have, $\sum_{i=1}^{n} \binom{d_i}{2} \leq \binom{n}{2}$. Since the function $f(x) = \binom{x}{2} = x(x-1)/2$ is a convex function, we can use Jensen's inequality to conclude that $\sum_{i=1}^{n} \binom{d_i}{2} \geq \frac{\binom{\sum_{i=1}^{n} d_i}{2}}{n} = \frac{\binom{2|E(G)|}{2}}{n}$ Thus, $\frac{\binom{2|E(G)|}{2}}{n} \leq \binom{n}{2}$ and therefore $|E(G)| \leq n^{3/2}$. □

Corollary 3.6. *If G is a thrackle with n vertices then $|E(G)| \leq n^{3/2}$.*

Proof. Since every thrackle has no C_4 subgraph, the assertion is true by the previous theorem. □

Notice that the previous corollary is still very far from Conway's thrackle conjecture. Now we try to obtain a better upper bound on the number of the edges of a thrackle. We need the following useful lemmas to obtain a $O(n)$ upper bound for the number of edges of a thrackle.

Lemma 3.7. *Let C_1 and C_2 be two closed curves (possibly self-intersecting) that do not touch each other. The number of intersections of C_1 and C_2 is an even number.*

Proof. Consider the closed curve C_1. This curve divides the plane into regions and these regions can be colored with the colors black or white so that every two adjacent regions have different colors. Now, if we consider the other curve, it crosses the boundary of every region even number of times. Since the black regions have no edge in common and also the boundary of the black regions cover the whole C_1 therefore the number of intersection points of C_1 and C_2 is precisely the same as the number of intersection points of C_2 and the boundary of black regions which is a sum of even numbers. This completes the proof of the lemma. □

Lemma 3.8. *Every graph G has a bipartite subgraph H so that $|E(G)| \leq 2|E(H)|$.*

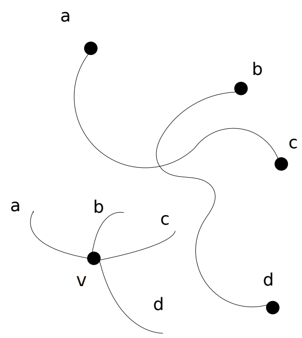


Figure 3.3: *A subdivision of K_5*

Proof. Let H be a bipartite subgraph of G with the maximum number of edges. Without loss of generality we can assume that H has all the vertices of G. Let A and B be a bipartition of the vertices of H. Let v be an arbitrary vertex of G. Assume that $v \in A$. If we consider the bipartite graph induced by the sets $A - \{v\}, B$, it cannot have more edges than the graph H because of the way we have chosen H. This means that in the graph G, v has more neighbors among the vertices in B compared to the vertices of A. So, the degree of v in H is at least half the degree of v in G. This argument is valid for every vertex v. Therefore H has the property that each of its vertices has degree at least half the degree in the graph G. Therefore $|E(H)| \geq |E(G)/2|$ □

Theorem 3.9. *Every bipartite thrackle graph is planar.*

Proof. First we prove that if G is a bipartite thrackle then G does not contain any subdivision of K_5. Suppose, in contrary, that G is bipartite thrackle with a subdivision of K_5. Let H be a subdivision of K_5 in G. Clearly H is also a bipartite thrackle graph. Let v be a vertex of H of degree 4 (notice that H has 5 vertices of degree 4 and the other vertices are of degree 2). Suppose that the neighborhood of v is as shown in Figure 3.3. Let C_1 be the closed curve produced by the paths $v-a$, $a-c$ and $c-v$. Let C_2 be the closed curve induced by the paths $v-b$, $b-d$ and $d-v$. These two closed curves are subgraphs of H and since H is a bipartite graph, each of them is induced by even number of edges. Since H is a thrackle, every edge of C_1 must cross every edge of C_2. On the other hand, Lemma 3.7 ensures that C_1 and C_2 will cross an even number of time. This is a contradiction since C_1 and C_2 intersect even number of times by their edges and one more time at the point v. It can also be shown that no subdivision of $K_{3,3}$ form a bipartite thrackle and therefore G has no subdivision of K_5 or $K_{3,3}$. Thus G is a planar graph. □

A graph drawn in the plane is called an odd thrackle if every two non adjacent edges cross odd number of times and every two adjacent edges cross even number of times (except the vertex they share). The above theorem is still true if we replace “thrackle” with “odd thrackle”. In fact it can be shown that a bipartite graph G is planar if and only if it is an odd thrackle.

Now, we are able to prove the following upper bound for the number of edges of a thrackle.

Theorem 3.10. *If G is a thrackle then* $|E(G)| \leq 4|V(G)|$

Proof. Suppose that G is a thrackle with n vertices. Let H be a bipartite subgraph of G with at least $\frac{|E(G)|}{2}$ edges. Lemma 3.8 guaranties that such a subgraph exists. Since H is bipartite and thrackle, it is a planar graph. Therefore $|E(H)| \leq 2n - 4$. Hence, $\frac{|E(G)|}{2} \leq 2n - 4$. $\square$

Chapter 4

Turán-type problems

4.1 Crossing edges, disjoint edges

Theorem 4.1 (Hopf-Pannwitz, 1934). *The maximal number of times that the diameter (the largest distance) can occur among n points in the plane is n for every $n \geq 3$.*

It is easy to show that in the geometric graph formed by diameters there are no two disjoint edges. We can prove a more general statement.

Theorem 4.2. *Let G be a geometric graph with no 2 disjoint edges. Then $|E(G)| \leq |V(G)|$.*

Proof. (Perles) We call a vertex $v \in V(G)$ *pointed* if all edges incident to v lie in a half-plane bounded by a line passing through v. For each pointed vertex v, delete from G the "leftmost" edge incident to v, that is, the first element in the clockwise order of edges around the half-plane at v. Notice that we deleted all edges, which proves the theorem. Indeed, if we are left with an edge $uv \in E(G)$, then it was not leftmost at u, nor at v. Thus, originally G contained two edges, uv' and $u'v$, with $\angle v'uv < \pi$ and $\angle u'vu < \pi$. These two edges must lie on opposite sides of the line uv. Hence, they are disjoint, contradicting the condition. □

In other words, the number of edges in a straightline thrackle is at most n.

Question. *Fix a $k \geq 2$. What is the maximum number of edges that a geometric graph on n vertices can have without containing k pairwise disjoint edges?*

Denote this number by f_k. The following table summarizes the current knowledge.

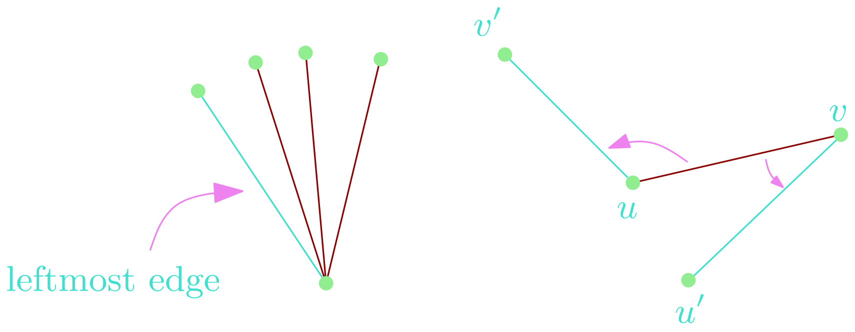


Figure 4.1: *Perles' argument*

$k = 2$	n	Hopf-Pannwitz, 1934
$k = 3$	$2.5n + O(1)$	Černý, 2003
$k = 4$	$\leq 8.5n$	Goddar-Kleitman, Katchalski
$k > 4$	$\leq 2^{10}(k-1)^2 n$	Pach-Törőcsik, Tóth 2000

A graph which has a drawing in the plane in which there is no k pairwise intersecting edges is called k*-quasiplanar*. Planar graphs are 2-quasiplanar and we know that they have at most $3n - 6$ edges. The following table gives the known bounds on the number of edges of k-quasiplanar graphs.

$k = 2$	$3n - 6$	Euler
$k = 3$	$6.5n - O(1)$	Agarwal et al., Ackerman-Tardos 2007
$k = 4$	$72(n-2)$	Ackerman 2006
$k > 4$	$O(n \log^{2k-8} n)$	Pach et al., Ackerman 2006
$k > 4$ (geom.)	$O(n \log n)$	Valtr 1997

A linear bound is conjectured.

Conjecture 4.3. *Every k-quasiplanar graph with n vertices has $\leq c_k n$ edges, where c_k is a constant depending on k.*

Definition 4.1. We call a geometric graph *convex* if its vertices are in convex position, i.e., they form a vertex set of a convex polygon.

Proposition 4.4 (Kuptiz, 1982)**.** *For any $n \geq 2k + 1$, the maximum number of edges that a convex geometric graph with n vertices can have without containing $k + 1$ pairwise disjoint edges is kn.*

Proof. Without loss of generality we can suppose that the vertex set of G is the set of vertices of a regular n-gon. Partition the set of segments (edges) into n classes so that two segments belong to the same class iff they are parallel. If G has no $k+1$ pairwise disjoint edges, then each class contains at most k elements of $E(G)$. Thus, $|E(G)| \leq kn$.

To show that the bound cannot be improved, take a graph G with vertices $V = \{x_0, \ldots, x_{n-1}\}$ and the edges $x_i x_{i+\lfloor n/2 \rfloor + j}$ $(0 \leq i \leq n-1,\ 1 \leq j \leq k)$. □

Theorem 4.5 (Capoyleas-Pach, 1992). *For any $n \geq 2k+1$, the maximum number of edges that a convex geometric graph with n vertices can have without containing $k+1$ pairwise crossing edges is*

$$2kn - \binom{2k+1}{2}.$$

The bound in the theorem is achieved for the following graph. Let $x_1, \ldots, x_n$ be the vertices of a convex n-gon in clockwise order. Connect x_i and x_j by an edge iff they are separated by fewer than k vertices along the boundary of the polygon, or $i \leq k$.

4.2 Extremal Graph Theory

Definition 4.2. By $ex(n, G)$ we denote the maximum number of edges that a graph with n vertices can have without containing a subgraph isomorphic to G. Analogously, for geometric and convex geometric graphs, we introduce $\overline{ex}(n, \overline{G}) \geq \overline{ex}_c(n, \overline{G})$ (here $\overline{G}$ denotes a geometric graph).

One of the most famous result in extremal graph theory is:

Theorem 4.6 (Turán).

$$ex(n, K_t) = \frac{n^2}{2}\left(1 - \frac{1}{t-1}\right) + O(n)$$

The following theorem is a useful generalization of Turan's theorem.

Theorem 4.7 (Erdős-Stone, Simonovits). *For any nonempty graph H*

$$ex(n, H) = \frac{n^2}{2}\left(1 - \frac{1}{\chi(H) - 1}\right) + o(n^2)\,,$$

where $\chi(H)$ denotes the chromatic number of H.

Theorem 4.8. $\overline{ex}(n, C_k) = ex(n, K_k)$, *where C_k denotes a non-selfintersecting geometric k-cycle.*

Proof. Let G be a geometric graph with n vertices and more than $ex(n, K_k)$ edges. Then G contains a complete geometric graph on k points. Every set of k points in the plane is the vertex set of a non-selfintersecting polygon. Therefore, there is an isomorphic copy of C_k in G.

On the other hand, it is easy to construct a convex geometric graph G with n vertices and $ex(n, K_k)$ edges that does not contain any simple closed polygon of length k. □

Definition 4.3. A geometric graph is called *outerplanar* if it can be obtained by adding some noncrossing internal diagonals to a simple closed polygon in the plane. Two geometric graphs are said to be *isomorphic* if they are topologically equivalent, that is, if there exists a one-to-one continuous mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that carries one into the other.

We can generalize the previous theorem to arbitrary outerplanar graphs.

Theorem 4.9 (Pach, Perles)**.** *Let $\overline{G}$ be an outerplanar graph i.e. a triangulated C_k. Then $\overline{ex}(n, \overline{G}) = ex(n, K_t)$*

All we need for the proof is the following lemma (the rest is the same as above).

Lemma 4.10 (Gritzmann-Mohar-Pach-Pollack)**.** *Let G be an outerplanar geometric graph with k vertices, and let P be any set of k points in the plane, no three of which are collinear. Then there exists a geometric graph G' isomorphic to G, with vertex set $V(G') = P$.*

Proof. We can assume that G is triangulated. We will prove a stronger statement: if x_1 and x_2 are two consecutive vertices of the closed polygon bounding G and p_1 and p_2 are two consecutive vertices of the convex hull of P, then we can find an assignment such that $x_1 \to p_1$ and $x_2 \to p_2$. The proof is by induction on k. For $k = 3$ it is easy. Let $k > 3$ and let $x_1, \dots, x_k$ be the vertices of G listed clockwise. Let x_i be the common neighbor of x_1 and x_2 in G. The following statement follows easily from a continuity argument: there exists a point $p \in P - \{p_1, p_2\}$ such that:

1. there are no points of P inside the triangle p_1p_2p;
2. there is a line l passing through p, separating p_1 from p_2 and not containing other points from P, such that there are exactly $i - 2$ elements of P in the open halfplane bounded by l which contains p_2.

Having this claim, we can proceed by induction. □

Proposition 4.11. *If $\overline{G}$ has a non-crossing cycle then $\overline{ex}_c(n, \overline{G}) \geq const \cdot n^2$.*

Proposition 4.12. *If $\overline{G}$ has a non-crossing Y then $\overline{ex}_c(n, \overline{G}) \geq const \cdot n^2$.*

Theorem 4.13 (Perles)**.** *Let $\overline{G}$ be a non-crossing caterpillar with k vertices. Then*

$$\overline{ex}_c(n, \overline{G}) = \left\lfloor \frac{n(k-2)}{2} \right\rfloor$$

Definition 4.4. The convex chromatic number $\chi_c(\overline{G})$ of a convex geometric graph $\overline{G}$ is the minimum number of colors in a good coloring such that the color classes consist of consecutive vertices.

Theorem 4.14 (Perles, Brass)**.** *If the convex chromatic number $\chi_c(\overline{G}) = t$, then*

$$\overline{ex}(n, \overline{G}) = \overline{ex}(n, K_t) + o(n^2)$$

Theorem 4.15 (Károlyi-Pach-Tóth 1997)**.** *For any 2-coloring of the edges of a complete geometric graph, one can find a non-crossing spanning tree all of whose edges are of the same color.*

Proof. By induction on n. If there are 2 consecutive edges of the convex hull of different colors, then we are done. So, we can assume that all edges along the convex hull are, say, blue. Denote the vertices by $v_1, \ldots, v_n$ with increasing x-coordinates. By induction for every point v_i we have a left monochromatic tree L_i and a right monochromatic tree R_i. We can assume that for every i the trees are of different colors, otherwise, we are done. Similarly, we can assume that R_2 and L_{n-1} are red. Now it follows that there is an i such that L_i and R_{i+1} are both blue. We can connect these trees by using an edge from the convex hull. □

Theorem 4.16 (Károlyi-Pach-Tóth 1997)**.** *For any 2-coloring of the edges of a complete geometric graph of $3n-1$ vertices, one can find n disjoint edges of the same color.*

Theorem 4.17 (Károlyi-Pach-Tóth-Valtr 1998)**.** *For any 2-coloring of the edges of a complete geometric graph of n vertices, one can find a monochromatic non-crossing path of length $\geq cn^{\frac{2}{3}}$.*

Question. *Does there also exist such a path of length $\geq cn$?*

Question. *What is the largest number $f(n)$ such that every complete geometric graph with n vertices has $f(n)$ pairwise crossing edges?*

Theorem 4.18 (Aronov et al. 1994)**.** $f(n) \geq \sqrt{n/12}$

Chapter 5

Ordnung muß sein!

5.1 Ramsey's Theorem

Theorem 5.1 (Ramsey 1930, Erdős-Szekeres)**.** *Every graph with n vertices has either a complete or an empty subgraph with $\frac{1}{2}\log_2 n$ vertices.*

Original theorem proposed by Ramsey in 1930 is more of a sociological statement than a mathematical theorem. Ramsey was a man of many interests, being born in high society and naturally curious, he proposed the following in puzzle form: amongst 6 people, one can always find three who mutually know each other or three that mutually do not know each other. In mathematical translation, this corresponds to: given K_6 and a 2-coloring of the edges, say red and blue. There exists either a red triangle or a blue triangle (or both).

Proof. Given K_6, pick any vertex. This vertex has degree 5. Given a 2-colouring of the edges, there must be at least three edges of the same colour, let this colour be blue. The vertices of the blue edges cannot be interconnected with blue edges amongst themselves, or else they would create a blue triangle. But since we have three of these vertices, and the edges interconnecting them must be red, there has to be a red triangle. $\square$

A more general statement is: for any k, there exists a smallest number $R(k)$ such that in any colouring of the edges of $K_{R(k)}$ by two colours, red and blue, we can always find a "monochromatic" K_k.

The function $R(k)$ has yet to be determined, the best known results are:

$$2^{k/2} \leq R(k) \leq 2^{2k},$$

with the upper bound being stated in the Theorem.

A more general form of Ramsey's theorem is the following:

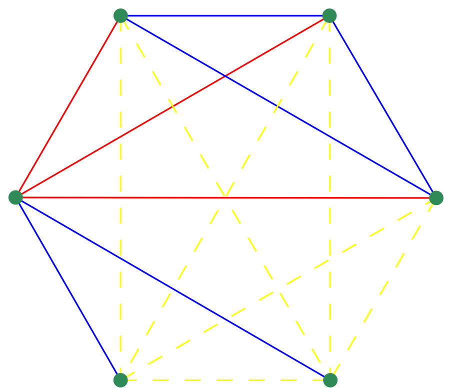


Figure 5.1: *Sketching the proof of the statement: Regard the leftmost vertex. It has degree 5 and at least three edges are of the same colour (red). These vertices cannot have a red edge between them as this would create a red triangle, but then a blue triangle appears.*

Theorem 5.2 (Ramsey)**.** *For any two positive integers r, s, there exists a least positive integer $R(r, s)$ such that for any complete graph on $R(r, s)$ vertices with blue and red edge colouring, there exists a complete subgraph on r vertices whose edges are all blue, or one on s vertices whose edges are all red.*

Proof. The proof is by induction on $r + s$. By definition, for all n, we have $R(n, 1) = R(1, n) = 1$. This proves the base case. Setting the induction hypothesis to $R(r - 1, s)$ and $R(r, s - 1)$ exist, we show that $R(r, s)$ exists.

We now claim that $R(r, s) \leq R(r-1, s)+R(r, s-1)$. Consider a complete graph on $R(r - 1, s) + R(r, s - 1)$ vertices. Pick a vertex v from the graph and partition the remaining vertices in two sets M, N, such that for every vertex w, w is in M if (v, w) is blue, and w is in N if (v, w) is red. Because the graph has $R(r - 1, s) + R(r, s - 1) = |M| + |N| + 1$ vertices, it follows that either $|M| \geq R(r - 1, s)$ or $|N| \geq R(r, s - 1)$. In the former case, if M has a red K_s then so does the original graph and we are done. Otherwise M has a blue K_{r-1} and so $M \cup v$ as a blue K_r by definition of M. The latter is analogous. Thus the claim is true and we have shown that $R(s, t)$ exists by induction. □

The following theorem was proved by Paul Erdős and George Szekeres in 1935 and is known as the Happy Ending Theorem.

Theorem 5.3 (Erdős-Szekeres)**.** *For a given pair of integers r, s, any sequence of length at least $(r - 1)(s - 1) + 1$ contains either a monotonically increasing subsequence of length r, or a monotonically decreasing subsequence of length s.*

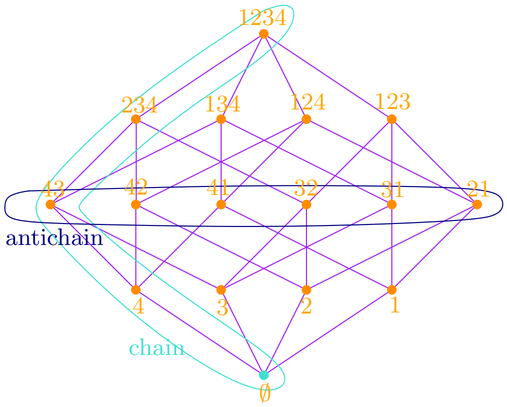


Figure 5.2: *An example of a comparability graph of the set* $1, 2, 3, 4$ *and all its subsets. Two elements are comparable if one is a subset of the other. A chain and an anti-chain are labelled green and red respectively.*

Proof. By contradiction. We label each number n_i in the sequence with a pair (a_i, b_i), where a_i is the length of the longest monotonically increasing subsequence ending with n_i and b_i is the length of the longest monotonically decreasing subsequence ending with n_i. By this construction, if we do not have a monotonically increasing or decreasing subsequence of the desired length, then $1 \leq a_i \leq r-1$ and $1 \leq b_i \leq s-1$ for all i. So there are $(r-1)(s-1) = rs - r - s + 1$ possible (a_i, b_i) pairs. But there are $rs - r - s + 2$ numbers in the sequence, so the pigeonhole principle guarantees that two numbers in the sequence, say n_j and n_k have the same pairs, which is impossible because assuming $j < k$, if $n_j < n_k$ we can append n_k to the monotonically increasing subsequence ending at n_j to get a longer monotonic sequence. If $n_j \geq n_k$, then we can append n_k to the monotonically decreasing subsequence ending at n_j to get a longer monotonic subsequence. In each case we have reached a contradiction, so the assumption that there is no monotonic subsequence of the desired length was false, hence proving the theorem. □

5.2 Dilworth's Theorem

If P is a set with a partial order defined on it, the *comparability graph* G of P has the vertex set P, with two vertices being adjacent iff the two elements are comparable. A *chain* in a partially ordered set is a subset which is totally ordered, i.e. whose elements are pairwise comparable. An *antichain* is a subset whose elements are pairwise incomparable.

An important theorem about partially ordered sets is Dilworth's theorem.

Theorem 5.4 (Dilworth 1950). *If the maximum length of a chain in a partially ordered set P is k, then P splits into k antichains. If the maximum size of an antichain is k, then P splits into k chains.*

Proof. We prove only the first part. For $x \in P$ let $r(x)$ denote the length of a longest increasing chain starting at x. By assumption we have that $1 \leq r(x) \leq k$ for all x. Let $X_i = \{x \in X : r(x) = i\}$. Since $X = \bigcup_i X_i$, it suffices to prove that X_i is an antichain for every i. Take $x, y \in X_i$. Suppose for a contradiction that $x < y$. We have

$$x < y = y_1 < y_2 < \ldots < y_i$$

for certain $y_2, \ldots, y_i \in P$, which implies that $x \notin X_i$. Contradiction. $\square$

Several interesting corollaries follow from Dilworth's Theorem. Some of those will only be mentioned briefly below.

Corollary 5.5. *Every partially ordered set of size $n > kl$ has either a chain of length $> k$ or an anti-chain of size $> l$.*

In particular, every partially ordered set of size n has either a chain or an antichain of size $\geq \sqrt{n}$. Notice that by a straightforward application of Ramsey's theorem to the corresponding comparability graph we get only a subset of size $\log n$ which is a chain or an antichain.

Corollary 5.6 (Gallai, Hajós). *Every system of $> kl$ intervals on a line has either $> k$ disjoint members or $> l$ intersecting members.*

Corollary 5.7 (Erdős-Szekeres). *Every sequence of $> kl$ reals has either an increasing subsequence of length $> k$ or a decreasing subsequence of length $> l$.*

Finally, we show a nice application of Dilworth's theorem to obtain a result in geometric graph theory. We are interested in the maximum number of edges that a geometrical graph with n vertices can have without containing $k + 1$ disjoint edges. We denote this number $ex(n, C_{k+1})$.

The special case where $k = 1$ is equivalent to straight line thrackles, which gives us $ex(n, C_2) = n$. We show now a non-trivial result that $ex(n, C_{k+1}) = O_k(n)$ for every fixed k. More precisely:

Theorem 5.8 (Pach-Törőcsik, 1994). *If the maximum number of pairwise disjoint edges in a geometric graph G is k, then $|E(G)| \leq k^4|V(G)|$.*

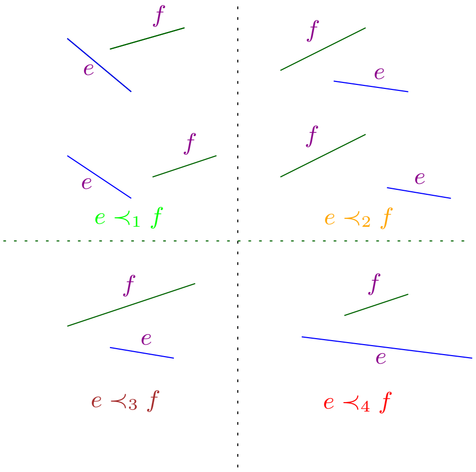


Figure 5.3: *Schematic definition of the four partial orders defined for our set.*

Proof. We start by defining four partial orders $\prec_1, \prec_2, \prec_3, \prec_4$ on the family of compact convex sets in the plane. Hopefully the reader will be satisfied with a pictorial definition (Figure 5.3).

An important property of the defined orders is that any two compact convex sets that are disjoint must be comparable with respect to at least one of the orders. Interpret $E(G)$ as the set of closed segments corresponding to the edges of G. Since G has no $k+1$ disjoint edges, this means that there is no chain of length $k+1$ with respect to $\prec_1$. By Dilworth's theorem, we can divide $E(G)$ into k subsets: $E_1 \cup \ldots \cup E_k$, that are anti-chains with respect to the first partial order. Now pick the largest of the subsets E_i, which is of size $\geq \frac{|E(G)|}{k}$, and further divide it into k subsets that are antichains with respect to the first two partial orders. One of the parts will have size $\geq \frac{|E(G)|}{k^2}$. Continue to divide in this fashion four times in total. We get in the end that there is $H \subset E(G)$ such that $|H| \geq \frac{|E(G)|}{k^4}$, and H is an antichain with respect to all four partial orders. It means that any two edges in H intersect at a point, i.e., H is a straightline thrackle. By a previous result, we know that $|H| \leq n$. Therefore, $|E(G)| \leq k^4 n$.

□

Definition 5.1. Let $f(n)$ denote the largest number such that any family of n convex sets in the plane has $f(n)$ disjoint or $f(n)$ pairwise intersecting members.

From the above proof we have $n^{1/5} \leq f(n)$. We now show a trivial construction for an upper bound. Consider $\sqrt{n}$ groups of $\sqrt{n}$ segments, so

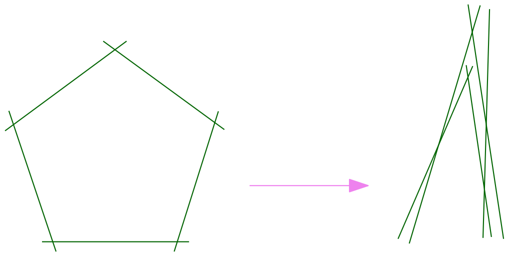


Figure 5.4: *Iterative construction for an upper bound on $f(n)$.*

that segments in each group are mutually intersecting, while the groups are pairwise disjoint. With this trivial construction we have a simple upper bound: $f(n) \leq \sqrt{n}$.

A less trivial construction tightens the bound to $\leq n^{0.42}$, where $0.42 \approx \frac{\log 2}{\log 5}$. The construction method is one which is commonly used in combinatorics and graph theory in general: we find a small configuration that is good and we iterate it. We create a pentagon out of five segments. With this configuration we have $f(5) < 2 < \sqrt{5}$. Now we replace each segment with what would look like a very thin squeezed pentagon as in Figure 5.4. Now we have 5^2 segments with the maximum number of intersecting (or disjoint) segments being 2^2. Iterating in this fashion we get $n = 5^k$ segments, with at most 2^k pairwise disjoint or intersecting segments. Since $2^k = n^{\log 2/\log 5} \approx n^{0.42}$, we have

$$n^{1/5} \leq f(n) \leq n^{0.42}$$

As one can see, the bounds are not quite tight. Further small refinements can be made, but this is more difficult.

Definition 5.2. Let $F_k(n)$ be the largest number such that any graph G that is a union of k comparability graphs $G_1, \ldots, G_k$ contains $F_k(n)$ vertices that form a complete subgraph or $F_k(n)$ vertices that are independent.

Similarly as above, we retain a lower bound for $F_k(n)$:

$$n^{\frac{1}{k+1}} \leq F_k(n)\,.$$

For a proof we simply color the edges in k different colors corresponding to the k partial orders. We have from previous results that we can find a portion of the graph of size $\geq n/l$ with no red edges. And in this subgraph we may find a subgraph of size $\geq n/l^2$ with no red or blue edges. After k steps we will find a portion of the original graph of size $\geq n/l^k$ with no edges of any color, meaning it is an independent set. So, for any l we have either a clique of size l or an independent set of size n/l^k. By taking $l = n^{1/(k+1)}$ we are done.

An upper bound was constructed too by Dumitrescu and Tóth:

Theorem 5.9 (Dumitrescu, Tóth)**.**

$$n^{1/(k+1)} \le F_k(n) \le n^{\log k/(k+1)}$$

This upper bound was constructed in a purely combinatorial way.

5.3 Davenport-Schinzel Sequences

With the development of computers, and especially computer graphics, the following problem arose: Given a camera angle on a field (usually 3D) with objects. Which portions of the objects can be seen directly from the cameras angle of view. This problem is core to fast and efficient rendering/processing of 3D computer graphics, and was originally thought to be simple. As it turned out, this problem, even in 2D was particularly difficult to solve. Davenport and Shinzel attempted the problem in 2D first and the problem goes as follows: given n segments in the plane, what is the maximum number of subsegments comprising the upper envelope of them? We denote this number by $f(n)$.

It was initially conjectured that $f(n) = O(n)$ (Atallah, 1965, D.-Sch. 1985). The lower bound can be set trivially to n. The upper bound has been gradually reduced: $f(n) = O(n \log n)$, and then $f(n) = O(n \log^* n)$, where $\log^* n$ is the iterative logarithm function. However, the final answer was surprising (see the theorem below).

The underlying idea is the following: let us number the segments 1 through n and write down the sequence of the labels along the upper envelope from left to right. We obtain a sequence with the following properties:

1. no two consecutive numbers are equal;

2. there is no alternating subsequence of length 5, i.e., there is no subsequence of the form $a \dots b \dots a \dots b \dots a$ (where $a \neq b$).

The first property is trivial by the construction of Davenport-Schinzel sequences. The second can be easily shown by using the fact that two straight lines can intersect at most once. A sequence satisfying the above properties is called a *Davenport-Schinzel sequence* of order n. The question now is equivalent to: what is the largest length of a Davenport-Schinzel sequence of order n?

Theorem 5.10 (Hart-Sharir, 1984)**.** *The maximum length of a Davenport-Schinzel sequence of order n is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of Ackermann's function. The bound is tight.*

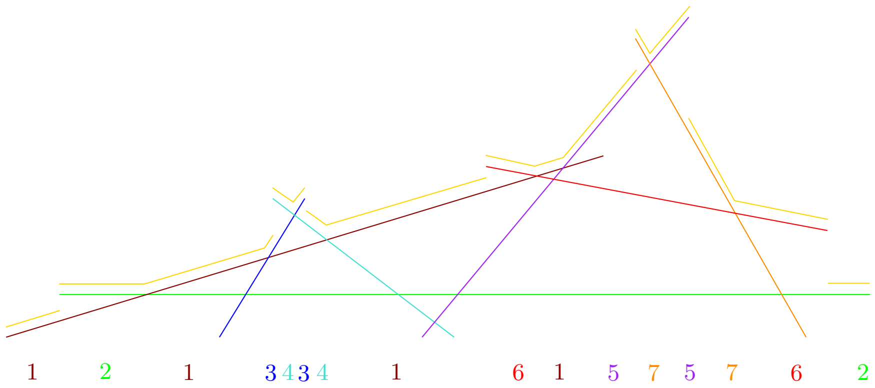


Figure 5.5: *Upper envelope of segments in the plane. If we label all segments and scan from left to right which segments the upper envelope is comprised of, we obtain a Davenport-Schinzel sequence.*

(The inverse Ackermann's function (even the iterative logarithm) has such a slow growth rate that, from a practical point of view, it can be considered as a constant. So, for all practical purposes, the upper bound can be seen as linear.)

Remark: The maximum length of a Davenport-Schinzel sequence with no alternating subsequence of length 4 is $2n - 1$.

Klazar and Valtr considered a natural extension of Davenport-Schinzel sequences by relaxing the second property:

Theorem 5.11 (Klazar-Valtr, 1994)**.** *Consider all sequences formed by n symbols such that any l consecutive elements are pairwise distinct, and there is no subsequence of type $a_1a_2 \dots a_{l-1}a_la_{l-1} \dots a_2a_1a_2 \dots a_{l-1}a_l$. For every $l \geq 2$, the maximum length of such a sequence of $O(n)$.*

Valtr used this theorem to derive the following result on geometric graphs.

Theorem 5.12 (Valtr, 1997)**.** *The maximum number of edges that a geometric graph of n vertices can have without containing k pairwise crossing edges is $O(n \log n)$.*

Proof. We can assume that the x-coordinates of the vertices are pairwise different and that the y-axis partitions $V(G)$ into two parts which are of equal size (more precisely, as equal as possible). Suppose we are able to show that the number of edges that intersect the y axis is $O(n)$. For the maximum number of edges $f_k(n)$ we now have that

$$f_k(n) \leq 2 \times f_k(n/2) + O(n) \, ,$$

which implies $f_k(n) = O(n \log n)$.

It remains to prove that $|E'| = O(n)$, where E' is the set of edges that cross the y-axis. Let $e_1, \dots, e_r$ be the edges in E' ordered in the order in which they cross the y-axis. Write on the left side of each crossing the label of the left endpoint of the corresponding edge and do the same thing on the right. In this way we get two sequences L and R of length $r = |E'|$ formed by n symbols. By Klazar-Valtr theorem the following two claims suffice to finish the proof.

Claim 1: For each $l \geq 1$ at least one of the sequences L and R contains a subsequence of length at least r/l^2 in which any l consecutive symbols are pairwise distinct.

Claim 2: Neither of sequences L and R contains a subsequence of type up-down-up$(R(k)^2)$, where $R(k)$ is the kth Ramsey number.

The first claim is proved as follows. We build two new sequences L' and R'. We start with empty sequences. Now we process the edges $e_1, \dots, e_r$ one by one, at the ith step we add the left endpoint of e_i to L' if it does not violate regularity and the right endpoint of e_i to R' similarly. It cannot happen that we do not add anything in l^2 consecutive steps. Therefore, the longer list will have at least r/l^2 elements at the end.

As for the second claim, we proceed by contradiction. We start with a long up-down-up subsequence and by Erdős-Szekeres we extract from it a long subsequence in which the x-coordinates of the corresponding vertices in the second "down" part are monotone. Let the sequence of vertices be $v_1, \dots, v_{R(k)}$ and the corresponding edges $e_1, \dots, e_{R(k)}$. Consider the following 2-colored complete graph: the vertices are $v_1, \dots, v_{R(k)}$ and the edge $v_i v_j$ with $x(v_i) < x(v_j)$ is blue iff v_j lies below the edge e_i, otherwise it is red. By Ramsey's theorem there is a monochromatic complete subgraph on k vertices. However, this implies that one can find k pairwise crossing edges in G. Contradiction. $\square$

Chapter 6

Beyond planarity

6.1 The Crossing Lemma

We recall the definition of the crossing number of a graph that was also mentioned before.

Definition 6.1. The **crossing number** of a graph G, denoted by $cr(G)$, is the smallest possible number of crossing in a drawing of G in the plane. Here we consider drawings with not necessary straightline edges, such that no three edges cross at a point.

Lemma 6.1. $cr(G) \geq e(G) - (3n - 6)$

Proof. By induction on $e(G)$. The basis is $e(G) = 3n - 5$ (for smaller $e(G)$ the statement is trivial). Since all planar graphs have at most $3n - 6$ edges, $e(G) = 3n - 5$ implies that G is not planar, i.e., $cr(G) \geq 1$. For the inductive step, take a graph G and its drawing with $cr(G)$ crossings. Remove an edge that forms at least one crossing and apply the hypothesis to the remaining graph. □

Theorem 6.2 (The Crossing Lemma)**.** *If G is a graph with n vertices and e edges and $e \geq 4n$, then*

$$cr(G) \geq \frac{1}{64}\frac{e^3}{n^2}.$$

Proof. Consider a drawing of G with $cr(G)$ crossings. We choose a random subset $V' \subset V(G)$ by including each vertex independently with probablity p. Let G' be the subgraph induced by V'. Denote by x the number of crossings in the drawing of G' induced by the drawing of G ($x \neq cr(G')$ in general). We have that

$$E[|V'|] = np, \qquad E[|E(G')|] = ep^2, \qquad E[x] = cr(G)p^4.$$

However, by the lemma we know that $x \geq e - 3n$ and, therefore, $cr(G)p^4 \geq ep^2 - 3np$. Setting $p = 4n/e$ (which is at most 1) we get

$$cr(G) \geq \frac{e^3}{64n^2}\,.$$

□

The order of magnitude of the above lower bound cannot be improved. To see this, take a graph G consisted of $n^2/(2e)$ cliques as equal as possible. Then in each clique there are $\Omega(e/n)$ vertices and the number of crossings is $\Omega(e^4/n^4)$ (by the crossing lemma). Hence, $cr(G) = \Omega(e^3/n^2)$.

By a more delicate argument the Crossing Lemma can be proved with a better constant.

Theorem 6.3 (Pach-Tóth 1997)**.** *If G is a graph with n vertices and e edges and $e \geq 7.5n$, then*

$$cr(G) \geq \frac{1}{33.75}\frac{e^3}{n^2}\,.$$

The following construction shows that the constant $1/33.75 \approx 0.0296$ is not far from optimal. Suppose that $n \ll e \ll n^2$. Take for the vertex set the vertices of the $\sqrt{n} \times \sqrt{n}$ grid and connect two points by an edge iff their distance is at most $\sqrt{\frac{2e}{\pi n}}$. Then

$$cr(G) \leq \left(\frac{16}{27\pi} + o(1)\right)\frac{e^3}{n^2} \approx 0.06\frac{e^3}{n^2}\,.$$

Lemma 6.4. *The maximum number of edges in a graph with n vertices that can be drawn in the plane so that every edge crosses at most 1 other edge is* $4n - 8$.

Proof (sketch). Take a maximal plane subgraph of G, denote it by G'. The edges in $E(G) - E(G')$ are all split into two by some edge of G'. We call these halfedges. It is easy to prove by induction that in each face f of G' with $s(f)$ sides we can have at most $s(f) - 2$ halfedges. Using Euler's formula and its corollaries we are done:

$$\begin{aligned} e(G) &= e(G') + (e(G) - e(G')) \leq \\ &\leq e(G') + \frac{1}{2}\sum_f (s(f) - 2) = 2e(G') - f(G') \leq 4n - 8\,. \end{aligned}$$

For an optimal construction take a planar graph whose all faces are quadrilaterals and then add two diagonals of each quadrilateral. □

Denote by $e_k(n)$ the maximum number of edges in a graph on n vertices that can be drawn in the plane so that every edge crosses at most k other edges. The above lemma says that $e_1(n) = 4n-8$. It can be also proved that $e_2(n) = 5n - 10$, $e_3(n) = 6n - 12$, $e_4(n) = 7n - 14$. It can be conjectured that $e_k(n) = (k+3)(n-2)$. As a consequence we have:

$$cr(G) > (e - 3n) + (e - 4n) + (e - 5n) + (e - 6n) + (e - 7n) = 5e - 25n\,.$$

Theorem 6.5 (Spencer-Szemerédi-Trotter)**.** *The maximum number of unit distances determined by n points in the plane is $O(n^{4/3})$.*

Proof. (Székely) Draw a graph G in the plane in the following way. The vertex set is the set of given points. Draw a unit circle around each point; in this way, consecutive points on the unit circles are connected by circular arcs: these are the edges of a multigraph drawn in the plane. Discard the circles that contain at most two points, and for any two points joined by some circular arcs, keep precisely one of these arcs. Let G be the resulting graph. The number of edges of this graph is at most $O(n)$ less than the number of unit distances. The number of crossings of G in this drawing is at most $2n^2$, since any two circles intersect in at most 2 points. An application of the Crossing Lemma finishes the proof. □

6.2 Halving segments

Definition 6.2. Given a set of n points in the plane, a segment connecting two of them in called a **halving edge** if the number of points on its left side is the same as on its right side.

The problem is to determine the maximum possible number of halving edges of an n-element set. Lovász obtained a bound $O(n^{3/2})$, which was later improved by P. Steiger and Szemerédi to $O(n^{3/2}/\log^* n)$. The best known bound is $O(n^{4/3})$, first proved by Dey, 1998. First we present Lovász' approach.

Theorem 6.6 (Lovász, 1971)**.** *The maximum number of halving edges is $O(n^{3/2})$.*

Proof. It is easy to show that the geometric graph G formed by halving segments has the "antipodality" property, i.e. in the cone opposite to the cone determined by any two consecutive neighbors of a vertex v there must be at least one neighbour. Moreover, by applying the same property we conclude that there must be in fact exactly one neighbor in each such cone. Also, every vertex has at least one neighbour. It follows that the degree of every vertex is odd and the number of neighbors of v that are to the left of v differs from the number of right neighbors by 1 (we can assume that no two vertices lie on a vertical line).

The crucial observation is: *every vertical line intersects at most $n/2$ halving edges.* To show this start with a vertical line l that cross k halving segments. Without loss of generality suppose that l has at most $n/2$ vertices on its left. Start translating l to the left. The number of halving segments intersected by l changes only when l passes through a vertex and then it changes by exactly 1. After at most $n/2$ such changes all the vertices will be to the right of l, so l will not intersect any segment. We conclude that $l \leq n/2$, as needed.

To finish the proof project all the vertices on the x-axis. Denote the projection points by $x_1 < x_2 < \cdots < x_n$. Mark every $\sqrt{n}$th point. There are $\sqrt{n}$ marked points. The number of edges projected to intervals containing some marked point is $O(n\sqrt{n})$, since every vertical line intersects $O(n)$ edges. Also, the number of edges projected to intervals not containing a marked point is $O(n\sqrt{n})$. $\square$

The following remarkable identity combined with the Crossing Lemma gives an improvement on the Lovasz' bound.

Theorem 6.7 (Andrzejak et al., 1998)**.** *Let G be the geometric graph determined by the halving segments of n points in the plane in general position. Let $k(G)$ be the number of crossing in G. Then we have*

$$k(G) + \sum_{v \in V(G)} \binom{(d(v)+1)/2}{2} = \binom{n/2}{2}.$$

Proof. *(Only the idea)* Start with the given point system and move its points continuously so that at the end they are in convex position. In convex position we have $n/2$ halving edges and every two of them cross, so that the equality holds. By investigating finitely many changes that can happen during the process one can see that these changes do not affect the validity of the identity.

$\square$

Theorem 6.8. *The maximum number of halving edges is $O(n^{4/3})$.*

Proof. The above identity implies in particular that $cr(G) = O(n^2)$. However, by the Crossing Lemma either $e(G) = O(n)$ or $cr(G) = \Omega(e^3/n^2)$. In any case, $e(G) = O(n^{4/3})$. $\square$

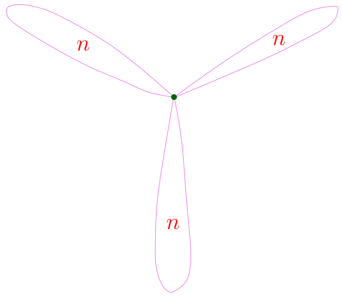


Figure 6.1: *Inductive construction for halving segments.*

As for the lower bound we show a construction with $\Omega(n \log n)$ halving segments. The construction is done by induction. Suppose we have a construction with n points and $h(n)$ halving edges. Then we can build a configuration with $3n$ points and $3h(n) + cn$ halving edges, for a certain constant $c > 0$. Indeed, we can squash the configuration of n points to make it look almost like a segment, we take three copies of it and arrange them as in Figure. We have $h(3n) \geq 3h(n) + cn$, which gives $h(n) = \Omega(n \log n)$.