Packing and covering

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9. Blichfeldt's density method

We have seen during the previous lectures that in the plane, the densest packing for any convex disc is a lattice packing. Thue's theorem provides the exact value when considering packings of unit circles. Naturally, we can ask if these results extend to higher dimensions, at least regarding packings of unit balls. In dimension 3, the answer to both of the questions is known to be true: in 1611, Kepler conjectured that a specific lattice packing of unit balls is the densest, with the corresponding density of $\pi/\sqrt{18}$. This conjecture remained unsolved until 1998, when Tom Hales gave a proof, using computer assisted methods. In fact, there are two different constructions providing the same density.

When the number of dimensions is at least 4, we do not know whether the densest packings are provided by lattice packings or not (in fact, there is little reason to hope that this holds in every dimension). Up to dimension 8, and in dimension 24, the densest lattice packings of unit balls have been determined. The latter case ($d = 24$) is known due to the existence of the Leech lattice, a structure having plenty of symmetries.

Several upper and lower bounds exist for the density of the densest packings of unit balls, however, there is a large gap between these. Finding lower bounds is achieved via constructions. The best lower bound is of order $c \cdot d \cdot 2^{-d}$, where c is an absolute constant. The best constant so far is due to Venkatesh from 2011. He also showed that in infinitely many dimensions, the lower bound can be strenghtened to $c \cdot log(logd) \cdot d \cdot 2^{-d}$.

When talking about upper bounds, the best one so far is the one obtained by Kabatjanskii and Levenstein, which is about $2^{-0.599d}$. In what follows, we present an older bound obtained by Blichfeldt in 1929, which is obtained by using an elegant method. The goal of Blichfeldt's method is to prove the upper bound $\frac{d+2}{2}2^{-d/2}$ for density of sphere packings in $\mathbb{R}^d$, for $d \geq 3$. The idea of the proof to approach the sphere packing analytically, by replacing each ball with a non-homogeneous density function. The packing criterion transforms to the condition that for any point of the space, the total density cannot exceed 1.

Before describing explicitly Blichfeldt's proof, we recall that any set $A \subset \mathbb{R}^d$ can be described using its characteristic function $\chi_A : \mathbb{R}^d \to \{0, 1\}$, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote the unit ball centred at the origin by B^d (thus, the unit ball centred at a point $c \in \mathbb{R}^d$ is $B^d + c$).

Given a set of centers $\mathcal{C} = \{c_1, c_2, ...\} \subset \mathbb{R}^d$ for the unit balls, one obtains that $B^d + \mathcal{C}$ forms a packing if and only if the following inequality holds:

$$\sum_{c \in \mathcal{C}} \chi_{B^d + c}(x) \leq 1.$$

Thus, from the geometric condition (disjoint spheres) we transformed the problem to an analytic setting. An important property of the characteristic function of the unit ball is that it is invariant under rotations, that is, it depends only on the Euclidean norm. Analytically, let us consider the function $\beta : \mathbb{R}_+ \to \mathbb{R}$, defined by

$$\beta(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}$$

Then each characteristic functions $\chi_{B^d+c_i}$ can be expressed by

$$\chi_{B^d+c} = \beta(|x - c|).$$

Thus, the inequality $\sum_{c\in\mathcal{C}} \chi_{B^d+c}(x) \leq 1$ is equivalent, in terms of β, to

$$\sum_{c\in\mathcal{C}} \beta(|x - c_i|) \leq 1.$$

Blichfeldt's idea was to replace the function β by a function D, which "blurs" the volume at the boundary, defined as follows:

$$D(r) = \begin{cases} 1 - \frac{r^2}{2} & \text{if } r \leq \sqrt{2} \\ 0 & \text{if } r > \sqrt{2} \end{cases} \tag{1}$$

The packing condition is replaced by the following criterion on the function D: for every packing of unit balls with centers $c_1, c_2, \ldots$, and for every $x \in \mathbb{R}^d$, we have

$$\sum_{c\in\mathcal{C}} D(|x - c_i|) \leq 1. \tag{2}$$

The following lemma provides an upper density of the packing.

Lemma 1. *Let $D(r) \geq 0$ be a function, which is continuous in the interval $[0, r_0]$ and vanishes for $r > r_0$. Suppose that for any packing $\{B^d+c_i | i = 1, 2, ...\}$ of unit balls and for any $x \in \mathbb{R}^d$,*

$$\sum_{c\in\mathcal{C}} D(|x - c_i|) \leq 1.$$

Then, the density of any packing of unit balls in $\mathbb{R}^d$ satisfies

$$\delta(B^D) \leq \frac{1}{d \int_0^{r_0} r^{d-1} D(r) dr}.$$

Proof. The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. $\square$

In order to check that the function $D(r)$ introduced above indeed satisfies the condition, we need the following lemma.

Lemma 2 (Blichfeldt's inequality)**.** *For any set of points $x, c_1, ..., c_n \in \mathbb{R}^d$, we have*

$$\sum_{i,j=1}^{n} |c_i - c_j|^2 \leq 2n \sum_{i=1}^{n} |x - c_i|^2.$$

Proof. Since translations do not change distances, we may assume that $x = 0$. In that case, the quantity on the right hand side above is simply $2n \sum |c_i|^2$. Recall that for any vector

$x \in \mathbb{R}^d$, the norm of x can be expressed by $|x|^2 = \langle x, x \rangle$. Hence,

$$\begin{aligned}\sum_{i,j=1}^{n} |c_i - c_j|^2 &= \sum_{i,j=1}^{n} \langle c_i - c_j, c_i - c_j \rangle \\ &= 2n \sum_{i=1}^{n} |c_i|^2 - 2 \sum_{i,j=1}^{n} \langle c_i, c_j \rangle \\ &= 2n \sum_{i=1}^{n} |c_i|^2 - 2 \left| \sum_{i=1}^{n} c_i \right|^2 \\ &\leqslant 2n \sum_{i=1}^{n} |c_i|^2.\end{aligned}$$

The above argument also shows that equality holds if and only if $x = \sum_{i=1}^{n} c_i$. □

We can give a nice physical interpretation to do the above result. Imagine that the points $c_1, \ldots, c_n$ are fixed, while x is moveable. Attach springs between c_i and x for every $i = 1, \ldots, n$. Then the total elastic energy stored in the system is proportional to $\sum_{i=1}^{n} |x - c_i|^2$. If we let the point x move, the system will find its equilibrium when this energy is minimized. The above Lemma expresses what this minimal energy is, moreover, it tells us that this is achieved when x moves to the centroid of the system of points $c_1, \ldots, c_n$.

We can now state the main result of this lecture, namely the upper bound for the density of unit balls in $\mathbb{R}^d$.

Theorem 1 (Blichfeldt)**.** *The density of any packing of unit balls in $\mathbb{R}^d$ is at most*

$$\frac{d+2}{2} 2^{-d/2}.$$

Proof. We have to check that the gauge function defined in 1 satisfies the condition 2. This is done easily by invoking Lemma 2, and using the fact that in any packing of unit balls, $|c_i - c_j| \geqslant 2$ holds for any pair of different centers c_i, c_j. After this, Lemma 1 provides us the bound invoking the integral, which can be easily calculated. The details can be found in J.Pach, P.Agarwal, Combinatorial Geometry. □

The table below presents some numerical results obtained using Blichfeldt's theorem (denoted by $b(d)$), compared with the densities of the densest lattice packings of B^d (denoted by $\delta_L(B^d)$), for values of d (the dimension) between 2 and 5.

d	$b(d)$	$\delta_L(B^d)$
2	1.000	0.907
3	0.883	0.741
4	0.750	0.617
5	0.618	0.465