Packing and covering

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8. Shadow cells

The last lecture focused on presenting the Voronoi cells. Without getting into details, we will mention a couple of properties, some of which will be discussed during the exercise sessions. First of all, any vertex of the Voronoi decomposition has the property that it is at equal distance from at least 3 points. Also, given a finite set of points in the plane, the only unbounded Voronoi cells will correspond to those points from the set lying on the boundary of the convex hull of the set of points.

Given the Voronoi decomposition of a set of points in general position, we can define also a dual decomposition in which all the faces are triangles and vertices are the n points. This is called the Delaunay triangulation. A more formal definition would be the following:

Definition. Let $c_1, ..., c_n$ be a set of n points in the plane. Let D_i be the Voronoi cell of c_i. Assume that no three points are on a line, and no four points are in the boundary of a circle. We will define T, the assigned Delaunay triangulation, as follows: T is the planar graph with vertex set $\{c_1, ..., c_n\}$ so that c_i and c_j are adjacent if and only if their corresponding Voronoi cells D_i and D_j are adjacent (i.e., if their boundaries intersect).

One can argue that the above defined Delaunay triangulation T is indeed a triangulation of $\text{conv}(c_1, ..., c_n)$, that is, the only face of T that is not a triangle is the outer face (this is the content of exercise 3 from problem set 8). Triangulations have several applications in mathematics as well as in various other fields; examples are the finite element method of partial differential equations, or computer image processing, 3D simluations, and so on.

Our purpose to introduce Voronoi cells was to estimate packing densities of convex cells. This works not only with the aid of Voronoi cells, but by using more general cell decompositions. The method is summarised in the following lemma.

Lemma 1 (Method of cell decomposition)**.** *Assume that $\mathcal{C} = \{C_1, C_2, ...\}$ is an arbitrary packing of convex discs in the plane and let $S_1, S_2...$ be cells such that, for all i, we have $C_i \subseteq S_i$ and for all $i \neq j$, $int(S_i) \cap int(S_j) = \emptyset$. If there exists $\delta, \Delta > 0$ such that $vol(C_i)/vol(S_i) \leq \delta$, $\forall i$ and $diam(S_i) \leq \Delta$ for every i, then the density if the packing is at most δ.*

In this lecture, we introduce another cell decomposition assigned to a packing: *shadow cells.* The goal, as before, is to provide bounds for the packing density of convex discs in the plane.

Definition (Shadow cells). Let $\mathcal{C} = \{C_1, C_2, ...\}$ be a packing of convex discs in the plane and let v be a nonzero vector. For every i, let S_i be defined as the set of those points $x \in \mathbb{R}^2$, which are either in C_i or, for which the intersection point of the ray parallel to v and starting at x with the set $C_1 \cup C_2 \cup ...$ belongs to C_i. S_i is called the **shadow cell** of C_i.

Using the shadow cell decomposition, we are going to give an elegant proof of the following theorem of Rogers.

Theorem 1 (Rogers, 1951)**.** *For any convex disc C in the plane, the density of an arbitrary packing of translates of this disc is at most $\delta_L(C)$, the maximal density of a lattice packing of translates of C.*

In brief, the above theorem states that the most efficient way to pack translated copies of a given convex disc is to use a lattice packing.

The proof that is presented in the book J.Pach, P.Agarwal, Combinatorial Geometry, is based on the following steps. First, we show that it suffices to prove the statement for centrally symmetric convex discs:

Lemma 2. *Given a convex disc Cin the plane, we define C^* as*

$$C^* = \frac{1}{2}(C + (-C)) = \left\{ \frac{c - c^{'}}{2} \mid c, c^{'} \in C \right\}.$$

Let $\mathcal{L}$ denote a set of vectors in the plane. Then $\mathcal{L} + C = \{\lambda + C | \lambda \in \mathcal{L}\}$ is a packing (of translates of C) if and only if $\mathcal{L} + C^* = \{\lambda + C^* | \lambda \in \mathcal{L}\}$ is a packing (of translates of C^*).

Thus, $\mathcal{L} + C$ is a packing with maximum density if and only of $\mathcal{L} + C^*$ is a packing with maximum density (plainly, because it has the most number of copies in a given circle with huge radius, and the number of copies is approximately the same for the two packings).

Next, we further reduce the problem to special cases, namely, to *trigonal discs*. Their definition follows:

Definition (Trigonal convex disc). A convex disc C is called *trigonal* if there exists a hexagon H with vertices $p_1, q_1, p_2, q_2, p_3, q_3$ in this order (say counter-clockwise), having its opposite sides parallel to each other, such that C is contained in H and the points p_1, p_2 and p_3 belong to the boundary of C. In this case, the points p_1, p_2, p_3 are called the vertices of C.

The reduction from centrally symmetric discs to trigonal discs is done by means of the following lemma:

Lemma 3. *For every centrally symmetric convex disc D in the plane, there exists a trigonal convex disc $C \subset \mathbb{R}^2$, such that*

$$\frac{C + (-C)}{2} = D.$$

Hence, by the same argument as before, the densest packing of D corresponds to the densest packing of C. Thus, it suffices to prove Rogers' theorem for trigonal discs. This is, in fact, not complicated; the argument is an intuitive argument, estimating the size of the shadow cells of members of the packing.

In essence, Rogers' theorem states that for convex discs, the most efficient packing is the lattice packing. A natural question arises: for which other domains does this property hold? Such shapes are called *Rogers domains*. There are only a few related results. On the one hand, there exists a construction which shows that not every planar shape is a Rogers domain. On the other hand, it is true that the union of two intersecting translates of a convex disc is always a Rogers domain.