Packing and covering

EPFL, 2014 Spring

7. The method of cell decomposition

We started off by re-stating and providing a new proof for Fáry's theorem (see the previous notes and the book). After this, we proceeded to the notion of difference regions.

Definition (Difference region). Given C a set, the difference region $D(C)$ is defined as

$$D(C) = C + (-C) = \{c - c' | c, c' \in C\}.$$

It was proven in the exercise session that $D(C)$ is always centrally symmetric and that if C is convex, then $D(C)$ is also convex. In what follows, we assume C to be convex (thus, $D(C)$ is convex as well).

Remember that we have defined an affine regular hexagon as the image of a regular hexagon under an affine mapping. An equivalent definition is that an affine regular hexagon contains 6 congruent copies of the same triangle.

The following lemma was listed as an exercise; the proof can be found in the boo k.

Lemma 1. *Every convex disc C has an inscribed affine regular hexagon. Moreover, if C is centrally symmetric, then there exists an inscribed affine regular hexagon with any prescribed direction of a chosen side.*

Next, we introduce the notion of Dirichlet-Voronoi cells corresponding to a set of points.

Definition (Dirichlet-Voronoi cells). Let D be a region in the plane and $O_1, \dots, O_n \in D$ be a set of points in the plane. For each i, we define D_i, the Dirichlet-Voronoi cell corresponding to the point O_i as the set of points for which O_i is the closest among the points $O_1, \dots, O_n \in D$; formally:

$$D_i = \{x \in D : \min_{1 \le j \le n} |x - O_j| = |x - O_i|\}.$$

Note that each boundary segment between neighbouring Voronoi cells is equidistant from two points of the given set. We also proved in the exercise session the following property regarding the Dirichlet-Voronoi cells: every cell is convex, bounded by some straight line segments (considering D above to be bounded) and pairwise disjoint. The following properties also hold.

Corollary 1. *Let us consider the above construction for the Dirichlet-Voronoi cells inside a bounded domain D and denote by s_i the number of sides of D_i. If D is a hexagon, then $\sum_{i=1}^n s_i \le 6n$.*

Corollary 2. *If each point O_i is the center of a unit disc C_i and the discs C_i form a packing, then $C_i \subseteq D_i$.*

Lemma 2. *Let $\mathcal{C} = \{C_1, \dots, C_n\}$ be a packing of unit circles in a bounded domain, and let $\{D_i\}$, $i = 1, \dots, n$ be the corresponding Voronoi cells. Then for every $i = 1, \dots, n$, we have*

$$\frac{A(C_i)}{A(D_i)} \le \frac{\pi}{\sqrt{12}}.$$

By using the above facts, we gave a new proof of Thue's theorem using Voronoi cells. All the proofs can be found in the book "Combinatorial Geometry".