# Packing and covering

EPFL, 2014 Spring

## 6. Coverings of $\mathbb{R}^2$ with congruent copies of a convex disc

In this lecture, we are going to estimate the density of coverings with congruent copies of a convex disc $C$. Our methods are similar to the ones used for estimating the density of packings; however, there are subtle difficulties that we have to handle.

Recall the basic lemma used in the case of packings: we considered a hexagon $H$ and a packing of any convex disc in $H$. The idea of the proof was to extend the discs to polygons, having in average at most 6 sides. What we need now is an analogous statement for coverings. However, the same method, namely, shrink the discs in order to achieve convex polygons $R_i$, for $1 \leq i \leq n$, does not always automatically work. As a counterexample, we provide the following figure (we consider $C_1$ to be the blue-colored shape and $C_2$ to be the red-colored one):

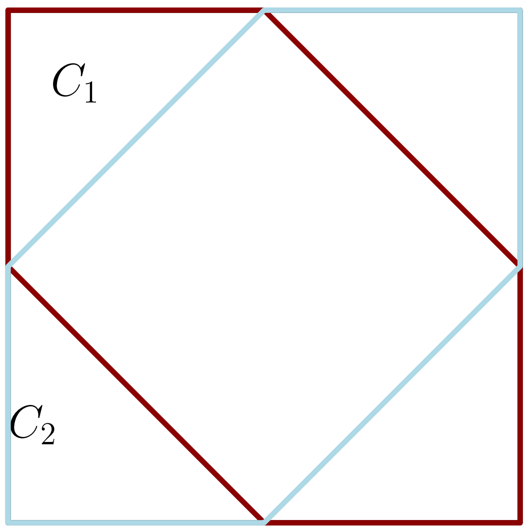


Figure 1

One can see that there is no way to cover $S = C_1' \cup C_2'$, where $C_1'$ and $C_2'$ are congruent copies of the same disc, and $C_1'$ and $C_2'$ do not cross. In order to avoid this situation, we will introduce the following definition

**Definition.** We say that two sets $C_1$ and $C_2$ are **crossing** if both of the sets $C_1 \backslash C_2$ and $C_2 \backslash C_1$ are disconnected.

**Remarks.**
(a) If two sets intersect in 2 points, then they are non-crossing.
(b) Under this assumption (namely that no two sets in the family cross), we can prove a dual to the lemma discussed above.

It was proven in the exercise session that if $C_1$ and $C_2$ are two translates of the same convex set $C$ in the plane, then they are non-crossing. We now state the lemma for coverings:

**Lemma 1.** *Let $C_1, ..., C_n$ be a system of non-crossing convex discs which cover a convex hexagon $H$. Then, one can find non-overlapping convex polygons $R_i \subseteq C_i$, for $1 \leq i \leq n$, which altogether cover $H$, and*

$$\sum_{i=1}^{n} s_i \leq 6n,$$

*where $s_i$ denotes the number of sides of $R_i$.*

*Proof.* The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. □

This implies the corresponding theorem for coverings:

**Theorem 1** (Fejes Tóth)**.** *Let $H$ be a convex hexagon in the plane. If $C_1, ..., C_n$ are congruent, non-crossing copies of a convex disc $C$ which completely cover $H$, then*

$$n \geq \frac{A(H)}{A(p_6)},$$

*where $p_6$ is a hexagon of largest area inscribed in $C$.*

*Proof.* The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. □

We have the following corollaries.

**Corollary 1.** *Let $H$ be a hexagonal region in the plane and $\mathcal{C}$ be a non-crossing covering of $H$ with congruent copies of $C$. Then*

$$d(H, \mathcal{C}) \geq \frac{A(H)/A(p_6)}{A(H)} \cdot A(C) = \frac{A(C)}{A(p_6)}.$$

Applying the above corollary for the case when $H$ gets larger and larger, we obtain that $\underline{d}(\mathbb{R}^2, \mathcal{C}) \geq \frac{A(H)}{A(p_6)}$.

Let $\mathcal{C}$ be a non-crossing covering of the plane with congruent copies of a convex disc $C$ and let

$$\vartheta^*(C) = \inf_{\mathcal{C}} d(\mathbb{R}^2, \mathcal{C}) = \min_{\mathcal{C}} d(\mathbb{R}^2, \mathcal{C}).$$

(it is not hard to prove that in fact the inf and min coincide). Obviously,

$$\vartheta^*(C) \leqslant \vartheta_L(C),$$

where by $\vartheta_L(C)$ we have denoted the minimal density of a lattice covering.

**Corollary 2.** *Let us consider the definitions in the corollary above. If $C$ is centrally symmetric, then*

$$\vartheta^*(C) = \vartheta_L(C) = \frac{A(C)}{A(p_6)}.$$

*Proof.* The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. □

**Corollary 3.** *For every centrally symmetric disc $C$, the density of the thinnest lattice covering with $C$ is*

$$\vartheta_L(C) \leq \frac{2\pi}{\sqrt{2\pi}}.$$

*Equality holds if and only if $C$ is an ellipse.*

Note that for lattice packings, the dual statement does not hold: it is not true among centrally symmetric convex discs, the circle provides the smallest lattice packing density. For an example, consider the "smooth octagon": it was proved that the density of any lattice packing using this figure is smaller than $\pi/\sqrt{12}$. We can also ask ourselves if there is any centrally symmetric disc that is "worse" in this respect than the smooth octagon or if there exists a non-centrally symmetric disc that is even worse. The answer to the second question is given by the following theorem:

**Theorem 2** (Fáry). *Let $C$ be a convex disc in the plane. Then, with the notation introduced before, the following inequalities hold:*

$$\delta_L(C) \geq \frac{2}{3} \text{ and } \vartheta_L(C) \leq \frac{3}{2}.$$

*These inequalities are tight when $C$ is a triangle.*