Packing and covering

EPFL, 2014 Spring

5. Packings with congruent or similar copies of a convex disc

Let us recall from the last lecture the following result:

Theorem 1. *If $\mathcal{C}$ is a packing of congruent copies of a convex disc C of the plane, then we have the following bound for the upper density of $\mathcal{C}$:*

$$\overline{d}(\mathcal{C}, \mathbb{R}^2) \leq \frac{A(C)}{A(P_6)},$$

where by P_6 we have denoted a hexagon of minimum area containing C.

As a corollary of this theorem, we obtained the following result:

Corollary 1 (Thue (1892, 1910))**.** *The density of any packing of unit circles in the plane is $\leq \frac{\pi}{\sqrt{12}}$.*

Remember that it was proven in the exercise sessions (problem set 4, exercise 5) that the following equality holds:

$$\sup_{\mathcal{C} \text{ packing of copies of } C} \overline{d}(\mathcal{C}, \mathbb{R}^2) = \max_{\mathcal{C} \text{ packing of copies of } C} d(\mathcal{C}, \mathbb{R}^2).$$

We shall denote this common value as $\delta(C)$. Thus, we can reformulate Thue's theorem using the notion of density of the densest packing with congruent copies of C instead of upper density. We will also define

$$\delta_L(C) := \sup_{\mathcal{C} \text{ lattice packing with copies of } C} \overline{d}(\mathcal{C}, \mathbb{R}^2).$$

Corollary 2. *For any centrally symmetric convex disc C,*

$$\delta(C) = \frac{A(C)}{A(P_6)}.$$

In order to prove this corollary, we will need the following lemma:

Lemma 1. *If C is a centrally symmetric disc in the plane, then P_6 is also centrally symmetric, where by P_6, we have denoted the hexagon of minimum area circumscribed about C.*

Proof (of the lemma). The proof of this lemma follows in fact from the proof of Dowker's theorem. Let us assume that P_6 is not centrally symmetric and consider $-P_6$. Since C is centrally symmetric, then $-P_6$ is also circumscribed about C. From the proof of Dowker's theorem, there is no improvement on areas, then we must have $P_6 = -P_6$, which completes the proof. □

Proof of Corollary 2. The inequality $\delta(C) \leqslant A(C)/A(P_6)$ follows directly from the theorem. We are only left to prove the inequality in the other direction. For this, we will use the above lemma. If P_6 is centrally symmetric, then the plane $\mathbb{R}^2$ has a lattice tiling with congruent copies of P_6: we start with P_6 and put congruent copies of it next to each other. We can easily observe that the centers of those copies of P_6 form a lattice and thus we obtain a lattice packing. We now inscribe in each copy of P_6 a copy of C, implying the inequality $\delta(C) \geqslant A(C)/A(P_6)$. □

Corollary 3. *For any centrally symmetric convex disc C, we have*

$$\delta(C) = \delta_L(C).$$

Remark. One could naturally ask if Corollaries 2 and 3 remain true for any convex disc? The answer is no. As a counterexample, just consider the case when C is a triangle. In this case, we have $\delta(T) = 1$, but $\delta_L(T) < 1$.

So far, we were only concerned with packings of congruent copies of a given convex disc. What happens, if we relax this condition to *similar* copies of a convex disc? If we allow the copies to be of arbitrary sizes, then the density of the packing can approach 1 as close as we wish. However, we can get a result similar to the above ones by imposing a condition on the sizes of the copies. This is the content of the following theorem of K. Böröczky and G. Fejes Tóth.

Theorem 2. *Let H be a convex hexagon, C be a convex disc and let P_s denote a convex s-gon of minimum area circumscribed about C. If $\mathcal{C} = \{C_1, ..., C_n\}$ is a packing of similar copies of C in H, and*

$$\frac{A(C_i)}{A(C_j)} \leq \frac{A(P_5) - A(P_6)}{A(P_6) - A(P_7)}, \text{ for all } i \text{ and } j,$$

then

$$d(\mathcal{C}, H) \leq \frac{A(C)}{A(P_6)}.$$

Proof. The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. □

Remark. One could ask if similar results also hold if we consider coverings instead of packings. A result in this direction is given by Kershner's theorem, which, informally, states that the density of the "thinnest" covering of the plane by unit circles is $2\pi/\sqrt{27}$.

Dual to the notion of packings, we shall define the following:

$$\vartheta(C) := \inf_{\mathcal{C} \text{ packing of copies of } C} \underline{d}(\mathcal{C}, \mathbb{R}^2) = \min_{\mathcal{C} \text{ packing of copies of}} d(\mathcal{C}, \mathbb{R}^2).$$

One could argue that the definition above is correct, namely that the minimum coincides with the infimum. This, in fact, can be done exactly in the same way as for density of packings.