# Packing and covering

EPFL, 2014 Spring

4. Packings with congruent copies of a convex disc

We have seen that the density of any lattice packing with congruent circles in the plane is at most $\pi/\sqrt{12}$. This lecture will focus on extending this result to any packing of congruent circles in the plane, by proving the following theorem of Thue:

**Theorem 1** (Thue (1892, 1910)). *The density of any packing of unit circles in the plane is* $\leq \frac{\pi}{\sqrt{12}}$.

We will see later in the course that the theorem above is in fact a corollary of another result, namely the theorem of Fejes Tóth. We are going to need the following definitions:

**Definition 1.** *Let $\mathcal{C} = \{C_1, C_2, \ldots\}$ be a collection of convex sets in the plane and $D \subseteq \mathbb{R}^2$ a region. We shall define the density of the collection $\mathcal{C}$ with respect to the region $D$ as*

$$d(\mathcal{C}, D) = \frac{\sum_{C_i \cap D \neq \emptyset} A(C_i)}{A(D)}.$$

*Let $Q_r = [-r, r] \times [-r, r]$. Let us define*

$$\underline{d}(\mathcal{C}, \mathbb{R}^2) := \varliminf_{r\to\infty} d(\mathcal{C}, Q_r),$$

$$\overline{d}(\mathcal{C}, \mathbb{R}^2) := \varlimsup_{r\to\infty} d(\mathcal{C}, Q_r).$$

*If the two numbers coincide, then we call the common value the* **density** *of the collection $\mathcal{C}$ in the plane and we will denote it by $d(\mathcal{C}, \mathbb{R}^2)$.*

**Remarks.**

(a) $d(\mathcal{C}, \mathbb{R}^2)$ does not necessarily exist! Take for example a collection of squares one inside of the other, the first one with side-length 10, the following $10^2$, the third one $10^3$, and so on. Fill the first square (the one in the middle) with the densest packing; leave the annulus between the second square and the first square empty; fill the annulus between the third and the fourth with the densest packing and so on. It is easy to see that the density of this packing does not exist.

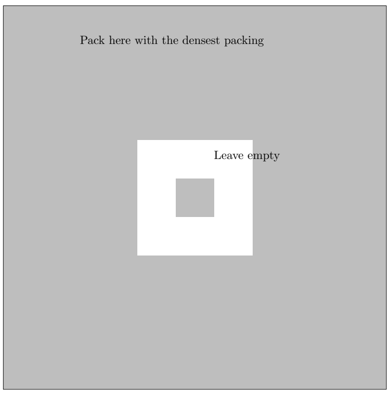


Figure 1

(b) if the density $d(\mathcal{C}, \mathbb{R}^2)$ exists, then it exists no matter what disc $D_r$ we choose in the definition instead of the square. Thus, we may define $D_r$ as $rD$, where $D$ is an arbitrary convex set containing the origin. Let us recall that $rD = \{rd : d \in D\}$.

**Lemma 1.** *Let $H$ be a hexagonal region and $C_1, ..., C_n \subseteq H$ a family of convex sets that form a packing. Then, we can find a system of non-overlapping convex polygons $R_1, ..., R_n \subseteq H$, such that $R_i \supseteq C_i$ for every $i$, and*

$$\sum_{i=1}^{n} s_i \leq 6n,$$

*where by $s_i$ we have denoted the number of sides of $R_i$.*

*Proof.* The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. $\square$

**Lemma 2** (Jensen's inequality)**.** *Let $a(x)$ be a real convex function and $x_1, ..., x_n \in \mathbb{R}$. Then*

$$a\left(\frac{x_1 + ... + x_n}{n}\right) \leq \frac{1}{n}\sum_{i=1}^{n} a(x_i).$$

**Theorem 2** (Fejes Tóth)**.** *Let $H$ be a convex hexagon and $C_1, \ldots, C_n$ a family of disjoint congruent copies of a convex disc $C$ packed in $H$. Then*

$$n \leq \frac{A(H)}{A(P_6)},$$

*where by $P_6$ we have denoted a hexagon of smallest area circumscribed about $C$.*

*Proof.* The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. $\square$

**Corollary 1.** *Given a packing $\mathcal{C}$ with congruent copies of a convex disc $C$ in the plane,*

$$\overline{d}(\mathcal{C}, \mathbb{R}^2) \leq \frac{A(C)}{A(P_6)},$$

*where by $P_6$ we have denoted the hexagon of minimum area circumscribed about $C$.*

In particular, taking $C$ to be a circle, $P_6$ is a regular hexagon circumscribed about $C$ and thus

$$\overline{d}(\mathcal{C}, \mathbb{R}^2) \leq \frac{\pi}{\sqrt{12}}.$$

This, in fact, provides a proof for Thue's theorem.