Packing and covering

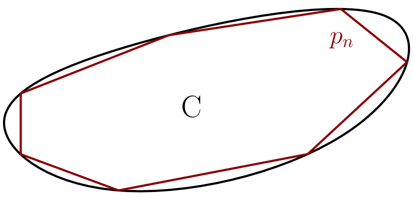
EPFL, 2014 Spring

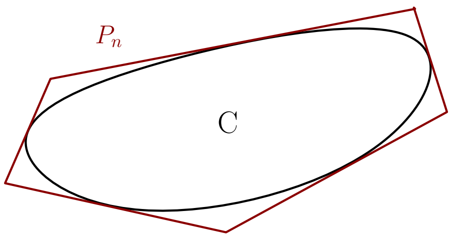
2. Approximating convex sets by polygons

A compact, convex set in the plane with nonempty interior will be called a **convex disc**. A special case is a **polygon**: this is the convex hull of finitely many points in the plane. For most applications, polygons are the easiest convex discs to deal with, and thus, it is very useful to approximate general convex discs by them. There are basically two models, which may be familiar from your previous studies, e.g. measure theory. For a convex disc C in the plane, we say that a polygon P is **inscribed** in C, if all the vertices of P lie on the boundary of C – then, by convexity, it follows that $P \subset C$. The inscribed polygons provide an inner approximation of C. For the outer approximation, we define **circumscribed** polygons: these are polygons P with the property that $C \subset P$, and all sides of P are tangent to C.

The area of polygons can be easily defined by taking a triangulation of them, that is, expressing them as the disjoint union of triangles. (One still has to check that this definition is consistent). Then, the area of an arbitrary convex disc C, denoted by $A(C)$, may be defined as the limit of the areas of inscribed polygons that approximate C arbitrarily well; this coincides with the limit of areas of well approximating circumscribed polygons. This is similar to the method of defining the Lebesgue measure of planar sets.

Let C be a planar convex disc, and let p_n be the inscribed n-gon (polygon with n sides) of maximum area. Similarly, let P_n denote the polygon with n vertices circumscribed about C of minimum area.





Theorem 1 (Dowker)**.** *Let C be a convex disc in the plane, $n \geq 3$, and let P_n denote the n-gon of minimum area circumscribed about C. Then*

$$A(P_n) \leq \frac{A(P_{n-1}) + A(P_{n+1})}{2}.$$

The proof can be found in J.Pach, P.Agarwal - Combinatorial Geometry.

There exists a dual to Dowker's theorem for inscribed polygons of maximum area.

Theorem 2 (Dowker's theorem for inscribed polygons)**.** *Given a convex disc C in the plane, $n \geq 3$, let p_n denote the n-gon of maximum area inscribed in C. Then*

$$A(p_n) \geq \frac{A(p_{n-1}) + A(p_{n+1})}{2}.$$

The proof of this result is analogous to the proof of Dowker's theorem; see the solution of the exercises.

Similar results can be proven if we consider perimeters instead of areas; we list these without proof.

Theorem 3 (Molnár, 1955; L.Fejes Tóth, 1959a)**.** *Let C be a convex disc in the plane, with $n \geq 3$. Let us denote by q_n the n-gon of maximum perimeter inscribed in C, and, respectively, by Q_n the n-gon of minimum perimeter circumscribed about C. We denote by* Per *the perimeter of a figure. Then, the following two statements hold:*

$$\operatorname{Per}(Q_n) \leq \frac{\operatorname{Per}(Q_{n-1}) + \operatorname{Per}(Q_{n+1})}{2}, \textit{ and}$$
$$\operatorname{Per}(q_n) \geq \frac{\operatorname{Per}(q_{n-1}) + \operatorname{Per}(q_{n+1})}{2}.$$

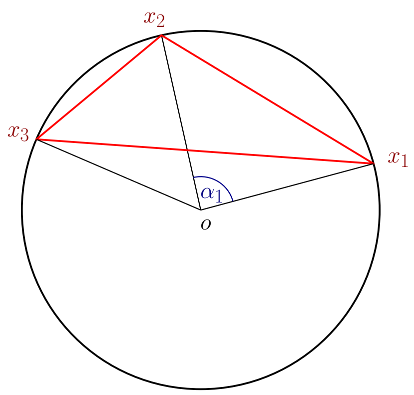
Dowker's theorem and its inscribed analogue involve the extremal area a circumscribed/inscribed polygon can have, without specifying this quantity. In special cases, we can calculate these areas.

Proposition 1. *The convex n-gon inscribed in the unit disc D with maximal area is the regular n-gon, whose area is $n/2\sin(2\pi/n)$.*

Proof. Let p_n be an n-gon inscribed in D, and denote its vertices by $x_1, \ldots, x_n$. Denote by $A(abc)$ the area of the triangle with vertices a, b and c. Let us consider three consecutive vertices of p_n, say, x_1, x_2 and x_3, and try to move x_2 so that the area of p_n becomes larger. We have

$$A(x_1 o\, x_2) + A(x_2 o x_3) = A(x_1 x_2 x_3) \pm A(x_1 o\, x_3),$$

where $+$ stands if $\widehat{x_1 o x_2} \leqslant \pi$ and $-$ stands if $\widehat{x_1 o x_2} > \pi$. In either case, to maximise the above area, we have to maximise $A(x_1 x_2 x_3)$, since $A(x_1 o\, x_3)$ is independent of x_2. It is easy to see that $A(x_1 x_2 x_3)$ is maximal if x_2 is exactly half-way between x_1 and x_3, thus, $\widehat{x_1 o x_2} = \widehat{x_2 o x_3}$. Since this condition must hold for every triple of consecutive vertices, we obtain that p_n must be regular.



In order to calculate the area of p_n, let α_i denote the angle $\widehat{x_i o x_{i+1}}$, where we set $x_{n+1} = x_1$. Then, by the sine formula for the area of triangles,

$$A(p_n) = \frac{1}{2} \sum_{i=1}^{n} \sin \alpha_i;$$

in the special case when p_n is regular, $\alpha_i = 2\pi/n$ for every i, and hence,

$$A(p_n) = \frac{n}{2} \sin\left(\frac{2\pi}{n}\right). \qquad \square$$

Thus, we obtain that the proportion between the area of the unit disc D and its best approximating inscribed n-gon is

$$\frac{A(p_n)}{A(D)} = \frac{n}{2\pi} \sin\left(\frac{2\pi}{n}\right).$$

The next result shows that this is the worst possible case. Note that affine transformations do not change this ratio, hence, the same bound holds for all ellipses.

Theorem 4 (Sas, 1939). *For any convex disc C, the area of the largest inscribed n-gon is at least*

$$A(C)\frac{n}{2\pi} sin\left(\frac{2\pi}{n}\right),$$

with equality if and only if C is an ellipse.

The proof can be found in J.Pach, P.Agarwal - Combinatorial Geometry.

One would naturally expect that the analogous statement holds for the outer polygonal approximations as well, namely, that the area of the best approximating circumscribed n-gon is the largest for ellipses. However, this is false! To see a counterexample, compare triangles circumscribed about a square and a circle (cf. Exercise 6.)