# Packing and covering

EPFL, 2014 Spring

## 11. Roger's bound on sphere packing - part 2

Remember that last time we have proved the following lemma, which is a consequence of Blichfeldt's lemma:

**Lemma 1.** *Let $\{B^d + c_i | i = 1, 2, ...\}$ be a packing and let $D(c_i)$ be the Voronoi-Dirichlet cell corresponding to $c_i$, $D(c_i) = \{x \in \mathbb{R}^d | \|x - c_i\| \leq \|x - c_j\|, \forall j\}$. Then, the distance from $c_i$ to any $d - k$-dimensional face of $D(c_i)$ is at least $\sqrt{\frac{2k}{k+1}}, \forall k = 1, ...d$.*

Note that $B + c_i$ is always contained in $D(c_i)$.

Let us now disect $D(c_1)$ into simplices of the form $conv\{v_0, v_1, ..., v_d\}$ recursively, as follows:

- consider $v_0$ to be $c_1$ (note that $c_1$ lies inside the polytope $D(c_1)$;

- divide $D(c_1)$: for each face, take the convex hull of this face, together with $c_1$. This decompose $D(c_1)$ as

$$D(c_1) = \bigcup_{F_{d-1} - (d-1) - dim.face} conv\{v_0, F_{d-1}\}.$$

- the next vertex $v_1$ is chosen as the point of $F_{d-1}$ the closest to $v_0$.

- once we have $v_1$, subdivide the face into simplices:

$$D(c_1) = \bigcup_{F_{d-1}} \bigcup_{F_{d-2} \subseteq F_{d-1}} conv\{v_0, v_1, F_{d-2}\}.$$

That means, at a given step, we have:

$$D(c_1) = \bigcup ... \bigcup conv\{v_0, v_1, ..., v_{k-1} F_{d-k}\}.$$

In this case, we have one of the two cases: either $d - k = 0$ (and we stop), or it is not (and in this case we choose $v_k \in F_{d-k}$ to be the closest to $v_0$.

Related to this decomposition, we can now state the following properties as a lemma:

**Lemma 2.** *The Voronoi-Dirichlet cell $D(c_1)$ can be decomposed into simplices of the form*

$$conv\{v_0, v_1, ..., v_d\},$$

*where $v_0 = c_1$, and*

*(i) $v_k$ lies in a $(d - k)$-dimensional face of $D(c_1)$ containing $v_k, v_{k+1}, ..., v_d$, and is the nearest point of this face to $c_1$ $(1 \leq k \leq d)$.*

*(ii) The scalar product*

$$\langle v_k - v_0, v_j - v_0 \rangle \geq \frac{2k}{k+1},$$

*for every $1 \leq k \leq j \leq d$.*

*Proof.* The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. □

Using the above lemmas, one can now prove Rogers' theorem (that gives the bound for sphere packing). The theorem is as follows:

**Theorem 1** (Rogers). *Let $S^d = conv\{p_0, ..., p_d\}$ be a regular simplex in $\mathbb{R}^d$, whose side length is 2. Draw a unit ball around each vertex of $S^d$. Let $\sigma_d$ denote the ratio of the volume of the portion of $S^d$, covered by balls, to the volume of the whole simplex, and let $\delta(B^d)$ be the density of the densest packing of unit balls in $\mathbb{R}^d$. Then*

$$\delta(B^d) \leq \sigma_d.$$

*Proof.* The proof can be found in J.Pach, P.Agarwal, Combinatorial Geometry. □

One can prove by some calculations that

$$\sigma_d = \left(\frac{1}{e} + o(1)\right) d \cdot 2^{-d/2}$$

as $d \to \infty$, showing that for large values of $d$, Rogers' bound is better than the bound of Blichfeldt by a factor of $2/e$.