Packing and covering

EPFL, 2014 Spring

1. Lattices

Let $u_1, u_2, ..., u_d$ be d linearly independent vectors in $\mathbb{R}^d$ (we will consider all of them being column vectors). We define the **lattice** generated by them as being:

$$\Lambda(u_1, ..., u_d) = \{a_1u_1 + ... + a_du_d,\ a_1, ..., a_d \in \mathbb{Z}\}.$$

The set $\{u_1, ..., u_d\}$ is called a **basis** of Λ. Observe that Λ is a discrete linear subspace of $\mathbb{R}^d$ with integer scalars, that is, a discrete set that is closed under addition and multiplication by integers. Also notice that every lattice in $\mathbb{R}^d$ is a non-singular affine image of $\mathbb{Z}^d$.

We will define the **fundamental cell/ fundamental parallelepiped** of the lattice Λ as being the set $P = \{\mu_1u_1 + ... + \mu_du_d,\ \mu_i \in [0, 1], 1 \leq i \leq d\}$. The fundamental parallelepiped has 2^d vertices, one of them being the origin. The lattice Λ may be obtained by tiling the space $\mathbb{R}^d$ with translates of P. Notice that for every point in Λ, the neighbouring vertices of the lattice span 2^d copies of P.

We know from linear algebra that the volume of the parallelotope P induced by the linearly independent vectors $u_1, ..., u_d$ equals to $|\det(u_1, ..., u_d)|$ (see Exercise 1.).

Given a lattice Λ, we define the **determinant of the lattice** as being the volume of the fundamental parallelepiped P; the following theorem will assure that this notion is well-defined, i.e. the volumes of the fundamental parallelotopes are all equal.

We say that Λ is a **unit lattice** if $\det \Lambda = 1$.

Theorem 1. *The determinant of the lattice is independent of the choice of the basis.*

Proof. Let Λ be our lattice. Consider two different bases of Λ: $(u_1, ..., u_d)$ and $(v_1, ..., v_d)$. We need to prove that $|det(u_1, ..., u_d)| = |det(v_1, ..., v_d)|$.

Since $(u_1, ..., u_d)$ is a basis of Λ, every vector $x \in \Lambda$ can be expressed as an integer combination of $(u_1, ..., u_d)$. In particular, since every v_i, $1 \leq i \leq d$, is a vector of the lattice Λ, it can be expressed as an integer combination of $(u_1, ..., u_d)$. Thus, we obtain:

$$\begin{gathered} v_1 = a_{11}u_1 + \ldots + a_{1d}u_d \\ \vdots \\ v_d = a_{d1}u_1 + \ldots + a_{dd}u_d \end{gathered}$$

with all $a_{ij} \in \mathbb{Z}$.

Introducing the $d \times d$ matrices

$$A = \begin{pmatrix} a_{11} & ... & a_{1d} \\ & ... & \\ a_{11} & ... & a_{1d} \end{pmatrix},\ V = (v_1 \ldots v_d),\ U = (u_1 \ldots u_d),$$

the above system of equations transforms to $V^\top = AU^\top$.

Similarly as above, since $(v_1, ..., v_d)$ also form a basis for the lattice, we can express every u_i as a linear combination of the vectors $v_1, ..., v_d$ with integer coefficients. Thus, we obtain

$$\begin{gathered} u_1 = b_{11}v_1 + \ldots + b_{1d}v_d \\ \vdots \\ u_d = b_{d1}v_1 + \ldots + b_{dd}v_d \end{gathered}$$

with all $b_{ij} \in \mathbb{Z}$.

Introducing

$$B = \begin{pmatrix} b_{11} & ... & b_{1d} \\ & ... & \\ b_{11} & ... & b_{1d} \end{pmatrix},$$

we obtain $U^\top = BV^\top$.

Considering the two identities $U^\top = BV^\top$ and $V^\top = AU^\top$, one arrives at $U^\top = BAU^\top$. Using that $\det(AB) = \det A \det B$ and that U is non-singular, we obtain that $\det(AB) = 1$. Note that since all the entries of the matrices A and B all integers, their determinants are integers as well. Thus, we deduce that $det(A) = det(B) = \pm 1$, whence $|det(U)| = |det(V)|$. Then $|det(u_1...u_d)| = |det(U)| = |det(V)| = |det(v_1...v_d)|$, which completes our proof. $\square$

Thus, any fundamental parallelepiped has the same volume. Moreover, it also follows that any parallelepiped of volume $\det \Lambda$ is a translate of a fundamental parallelepiped; and the same is true for any empty parallelepiped (see Exercise 3.)

A lattice vector u is called **primitive**, if there is no other lattice point on the segment between 0 and u. It follows from the above remarks that any set of linearly independent primitive lattice vectors can be completed to a basis of Λ.

During the course, we are mainly interested in the density of packings and coverings of the space. This may be defined as follows. Assume that S is an (infinite) subset of $\mathbb{R}^d$ (usually, S is the union of translates of a convex body). Then the **density** of S in $\mathbb{R}^d$ is given by the limit

$$\lim_{r\to\infty} \frac{\lambda(S \cap rB^d)}{\lambda(rB^d)},$$

where λ denotes the Lebesgue measure on $\mathbb{R}^d$, and B^d stands for the d-dimensional unit ball. Thus, if the density exists, it means the proportion of the space covered by S.

It is good news that the density of special arrangements related to lattices exists. Namely, let K be a convex body (a closed, bounded, convex set in $\mathbb{R}^d$). Let Λ be a lattice in $\mathbb{R}^d$. Consider the set of translates of K with the lattice vectors:

$$\{K + u \,:\, u \in \Lambda\}.$$

Assume that none of these translates overlap. Then the density of the set of translates exists, and it is exactly

$$\frac{\mathrm{vol} K}{\det \Lambda}$$

(see Exercise 5.) If the translates overlap, then the above fraction equals to the *covering density with multiplicities*, that is, the average of the number of times a point of $\mathbb{R}^d$ is covered by the translates of K.

The determinant of the lattice carries important information about Λ. The following two theorems show that there can not be "too big holes" in a lattice with small determinant.

Theorem 2. *Let Λ be a unit lattice in $\mathbb{R}^2$. Then, there are two points of the lattice whose distance apart is at most $\sqrt{2/\sqrt{3}}$.*

Proof. Let $u, v \in \Lambda$ be two points of the lattice whose distance is minimum (this minimum indeed exists, because by the translation invariance of the lattice, we may assume that $u = 0$, and therefore we only have to search v in a bounded neighbourhood of $u = 0$). Denote this minimum distance by δ^*. Assuming $u = 0$, it follows from the linearity of Λ that $kv \in \Lambda$ for every $k \in \mathbb{Z}$. Let denote by ℓ the line spanned by v.

Consider a circle of radius δ^* around each point of the kv. By the minimal distance property, there is no lattice point in the interior of these circles, apart from their centres. One can see that those circles will completely cover a strip of half-width $(\sqrt{3}/2)\delta^*$.

Let us consider P, a fundamental parallelogram of Λ with two vertices $u = 0$ and v. Since $\det \Lambda = 1$, the side of P opposite to uv is at distance $1/\delta^*$ from ℓ. By the above observation, it follows that half-width of the empty strip must be at most this quantity. This is equivalent to $(\sqrt{3}/2)\delta^* \leqslant 1/\delta^*$, which leads to $(\delta^*)^2 \leqslant 2/\sqrt{3}$. □

The regular triangular lattice shows that this bound is tight.

Corollary 1. *The density of a lattice packing of discs in* $\mathbb{R}^2$ *is at most* $\pi/\sqrt{12} \approx 0.906$.

Proof. Since the density of a packing is invariant under scaling, we may assume that Λ is a unit lattice. The largest possible radius so that the translates of the discs do not overlap is half of the minimum distance between non-identical lattice points. From Theorem 2 we obtain that the radius can be at most

$$r^* = \frac{1}{2}\sqrt{\frac{2}{\sqrt{3}}}.$$

Hence, the density of the lattice packing of congruent circular discs on the plane is at most

$$\frac{\text{Area(r*D)}}{\det \Lambda} = \frac{(r^*)^2\pi}{1} = \frac{\pi}{\sqrt{12}}$$ □

The next important result gives an upper estimate for size of empty convex regions in Λ. A set $C \subset \mathbb{R}^d$ is *centrally symmetric* if $C = -C$.

Theorem 3 (Minkowski)**.** *Let* $C \subset \mathbb{R}^d$ *be a centrally symmetric convex body, and let* λ *be a unit lattice. If vol*$(C) > 2^d$*, then* C *contains at least one lattice point different from* 0.

Proof. Let us consider the set of translates

$$\{(1/2)C + u,\, u \in \Lambda\}.$$

Since $\mathrm{vol}\,((1/2)\,C) = (1/2)^d \mathrm{vol}\,(C) > 1$, the density of this family of translates is strictly greater than 1, hence, they cannot form a packing. Thus, there exists $x \in \mathbb{R}^d$, which is contained in at least two translates: that is, there exist $u, v \in \Lambda$ with $u \neq v$, so that

$$x \in \left(\frac{1}{2}C + u\right) \cap \left(\frac{1}{2}C + v\right).$$

Thus, $x - u \in (1/2)C$ and $x - v \in (1/2)C$, and since C is centrally symmetric, it also follows that $v - x \in (1/2)C$. The convexity of C implies that the midpoint of these two is also contained in $(1/2)C$:

$$\frac{(x-u)+(v-x)}{2} = \frac{v-u}{2} \in \frac{1}{2}C,$$

which shows that $v - u \in C$ – but this is a non-zero point of the lattice Λ. □

Next, we give a neat proof to a classical result that is well familiar.

Theorem 4. *Let* n *and* m *be two integers with* $(m, n) = 1$*. Then there exist integers* a, b *so that*

$$a\,m - b\,n = 1.$$

Proof. Consider the lattice $\mathbb{Z}^2$. Geometrically, the condition $(m, n) = 1$ means that there is no lattice point on the segment between the origin and the point (m, n), that is, (m, n) is a primitive lattice vector. Hence, it can be completed to a basis: there exists a lattice point (b, a), so that (m, n) and (b, a) form a basis. The volume of the parallelepiped determined by these equals to 1; expanding the determinant, we obtain

$$1 = |am - bn|,$$

from which the assertion follows (maybe by taking $(-a, -b)$). □

Finally, without proof, we mention a useful result which allows us to calculate the areas of planar lattice polygons.

Theorem 5 (Pick)**.** *Let Λ be a unit a lattice on the plane, and let P be a simply closed lattice polygon, that is, a polygon all of whose vertices are points of Λ. Then*

$$\mathrm{Area}(P) = i + \frac{b}{2} - 1,$$

where i is the number of lattice points in the interior of P, and b is the number of lattice points on the boundary of P.