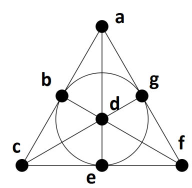
Discrete mathematics - problem set 9

November 17, 2016.

1. Let $a_1, ..., a_n \geq 1$ be fixed real numbers and $A = \{\sum_{i=1}^{n} \epsilon_i a_i,\ \epsilon_i = 0 \text{ or } 1\}$. At most how many of the 2^n sums in A can lie inside an interval I of length less than 1?

2. The "Fano plane" (see below) is a family of 3-element subsets of the set $X = \{a, b, c, d, e, f, g\}$ consisting of the following 7 triples: $\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{b, e, g\}, \{c, d, g\}, \{c, e, f\}$. In the figure, the points represent the elements of the base set, while the lines and the circle represent members of the set family.



   (a) Prove that the Fano plane is a *maximal* intersecting family of 3-sets, that is, no other subset of three points could be added in a way that the family remains intersecting.
   (b) What is the maximum number of 3-element subsets that a 7-element set can have with the property that no two of them are disjoint?

3. A family of sets is called *intersecting* if any two of its members intersect. Given a set X with $|X| = 15$, prove that the maximum size of an intersecting family of 8-element subsets of X is at least as large as the maximum size of an intersecting family of 10-element subsets of X.

4. Let $k \geq n/2$ and let $\mathcal{F}$ be a family of k-element subsets of $\{1, 2, ..., n\}$ such that $A \cup B \neq \{1, 2, ..., n\}$, for all $A, B \in \mathcal{F}$.
   (a) Prove that $|\mathcal{F}| \leq \binom{n-1}{k}$.
   (b) Show that the above bound is tight.

5. In a kindergarten there are 12 boys (3 of them are 3 years old, 5 are 4 years old and 4 are 5 years old) and 9 girls (4 of them are 3 years old, 2 are 4 years old and 3 are 5 years old). Assuming that each child is picked with equal probability:
   (a) What is the probability of picking a girl?
   (b) What is the probability of picking a girl, provided that we pick a 3-year-old?
   (c) What is the probability of picking a 3-year-old, provided that it is a girl?

6. Let σ be a permutation of $\{1, ..., n\}$ selected randomly from the set of all permutations. What is the expected value of the number of fixed points? Recall that i is a fixed point if $\sigma(i) = i$.

7. Let $(1, 2, ..., n)$ be a circular sequence (mod n), and let k be an integer such that $n/3 < k \leq n/2$. Prove that the maximum number of pairwise intersecting intervals on $(1, 2, ..., n)$, each consisting of k consecutive integers (mod n), is k.

8*. What is the maximum number m for which different subsets $X_1, \ldots, X_m$ of $\{1, \ldots, n\}$ exist with the property that $X_a \cap X_c \subseteq X_b$ for any $1 \leq a < b < c \leq m$?