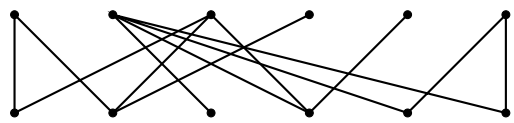
Discrete mathematics - problem set 7

November 3, 2016.

A matching in a graph is a set of disjoint edges (two edges are disjoint if they do not share any vertex). A matching M is called a perfect matching if every vertex in the graph is contained in an edge of M.

1. Does the following graph have a perfect matching? (Explain your work.)



2. Let $G = (A \cup B, E)$ be a bipartite graph, with all vertices in A having the same degree k, and all vertices in B having the same degree l, and $k > l$. Show that G has a matching M such that every vertex in A is contained in an edge of M.

3. The new 100 member congress has formed 20 congressional committees from its members. The same congressman can belong to several committees. Each committee needs to have a chairperson. Find a necessary and sufficient condition that the these chairpersons can be selected in such a way that no-one is the chairperson of more than one committee.

4. Let $G = (A \cup B, E)$ be a bipartite graph such that every vertex of G has degree at least one. A set of vertices X is called an *independent set* if no edge of G has both endpoints in X.

   Prove that the maximum size of an independent set in G is equal to the minimum size of an edge set F in G such that every vertex in G is the endpoint of some edge in F.

5. Let $G = (A \cup B, E)$ be a bipartite graph, such that $|A| = |B| = n$. Prove that G contains a matching of size $n-1$ if and only if for any subset Y of A, we have $|N(Y)| \geq |Y| - 1$ (where $N(Y)$ denotes the set of neighbors of Y in B).

6. Prove that a graph is bipartite if and only if it does not contain a cycle of odd length.

7. Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = n$. Prove that if G has minimum degree at least $n/2$ then it contains a perfect matching.

8*. Big Square country occupies a large round island. The 144 cities of the country are arranged in a 12×12 square grid (see below). One day the country is invaded by the Peacefuls. The invaders decide to separate the cities of Big Square country by digging straight canals across the island (from sea to sea) in such a way that every resulting piece of land contains exactly one city. Help the invaders: can they do this with fewer than 22 canals?

   What if the cities are arranged in another pattern: is there a pattern where not even 70 canals are enough?