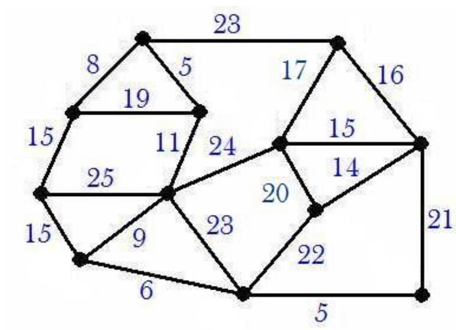
Discrete mathematics - problem set 5

October 20, 2016.

1. Let G be a connected weighted graph where the weights are not necessarily different.
   (a) Prove that for any edge e of minimum weight, there is a minimum spanning tree containing e.
   (b) Let v be a vertex in G and suppose that among the edges touching v, e has a unique smallest weight. Prove that e is contained in any minimum spanning tree in G.
2. Apply Kruskal's algorithm to the following graph to obtain a minimum spanning tree:



3. Prove that Kruskal's algorithm works correctly also in the case when two or more edge weights are equal, if in each step the algorithm picks one of the lowest-weight edges randomly.
4. Does Kruskal's algorithm find a minimum spanning tree if the weights are allowed to be negative?
5. (a) Let G be a connected graph with weighted edges, and let U be an arbitrary subset of the vertices. Suppose e is of minimum weight among the edges with exactly one endpoint in U. Prove that G has a minimum spanning tree that contains e.
   (b) The following is called Prim's algorithm:
   - *Initialize a tree with a single vertex, chosen arbitrarily from the graph.*
   - *Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and add it to the tree.*
   - *Repeat step 2 (until all vertices are in the tree).*

   Prove that, given a graph G as input, the output of Prim's algorithm is a minimum spanning tree of G. Apply Prim's algorithm to the graph from exercise 2.
6. Consider an n-vertex complete graph with a selected edge e. Prove that the graph contains $2n^{n-3}$ spanning trees containing the edge e.

7*. A magician has 50 cards on a table, numbered $1, 2, \ldots, 50$. In a step, you can remove two arbitrary cards, write the difference of their values on a new card, and put this new card back on the table (and discard the other two).
The magician hides behind a curtain while you take 49 such steps, and then claims (correctly) that the number on the remaining card is not 0. Prove that he did not have to peek to make that guess because there was no way the last card would have 0 on it.