Discrete mathematics - problem set 3

October 6, 2016.

1. How many positive integers under 574 are *not* divisible by either of the numbers $2, 3, 5, 7$?

2. (a) Find the number of ways 6 people can sit around a round table.
   (b) Determine the number of ways 3 couples can sit around a round table in such a way that no couple sits next to each other.

3. Determine the number of permutations with exactly one fixed point.

4. A function $f : X \to Y$ is called *surjective* or *onto* if for any $y \in Y$ there is some $x \in X$ such that $f(x) = y$. Show that the number of surjective functions $f : [m] \to [n]$ is

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^m.$$

5. For a vertex v in a graph, we define the degree $d(v)$ to be the number of edges touching v. Show that the sum of the degrees is always equal to two times the number of edges.

6. Show that every finite graph on at least two vertices contains two vertices of equal degree.

7. Let G be a graph on n vertices. Prove that the followings are equivalent:
   a graph is acyclic if it contains no cycle as a subgraph
   (a) G is connected and acyclic (i.e., it is a tree)
   (b) G is connected and has $n - 1$ edges
   (c) G is acyclic and has $n - 1$ edges.

8 ***.** Uncle John has n buckets of apples. He wants to distribute some of them evenly among his n nephews and nieces. Prove that he can select some (at least one) of those buckets so that the total number of apples in them is divisible by n.

Note: Read the section about the Euler function in the Matoušek-Nešetřil book (pages 105-106)