Discrete mathematics - problem set 10

November 26, 2015.

1. Which of the following statements are true for a probability space $(\Omega, \mathbb{P})$?
   (a) If A, B are disjoint events (they cannot occur at the same time) then they are independent.
   (b) Ω and $A \subset \Omega$ are independent.
   (c) If A and B are independent then A and $\overline{B}$ are also independent.

2. A random graph $G(n, p)$ is a probability space, where for each pair $1 \leq i < j \leq n$, (i, j) is an edge of $G(n, p)$ with probability p, independently of any other edge (you can think of a sequence of independent coin tosses for each edge). Compute the following:
   (a) the expected number of edges in $G(n, p)$
   (b) the expected degree of a vertex in $G(n, p)$
   (c) the expected number of triangles (cycles of length 3) in $G(n, p)$
   (d) the probability that the degree of a given vertex v is at most k

3. Let $X_1, \ldots, X_n$ be independent random variables such that for every i, $X_i = 1$ with probability p and $x_i = 0$ otherwise. Let $X = \sum_{i=1}^{n} X_i$. What is the expectation of X^2?

4. Let G be a graph with m edges, and let k be a positive integer. Prove that there is a coloring of the vertices of G with k colors so that at most m/k edges of G connect two vertices with the same color.

5. Let $\mathcal{F}$ be a family of 3-element subsets of a set X. Prove that the elements of X can be colored with 3 colors so that at least $|\mathcal{F}| \cdot 3!/3^3$ sets in $\mathcal{F}$ have exactly one element of each color.

6. In the lecture we gave a probabilistic argument that shows that any graph G on m edges contains a bipartite subgraph that has at least $m/2$ edges. Show that the same argument actually proves the existence of a bipartite subgraph that has *strictly* more than $m/2$ edges.

7. Consider the following algorithm on the graph $G = (V, E)$. First split the vertices arbitrarily into two parts V_1 and V_2. Then in each step, if there is a vertex $v \in V_i$ that has more neighbors in V_i than in V_{3-i}, then move v to V_{3-i}. Prove that this algorithm stops in a finite number of steps, and at the end the bipartite subgraph between V_1 and V_2 contains at least half of the edges of G.

8*. After figuring out his last trick, you decide to pay your favorite magician another visit. This time he asks you to think of n arbitrary distinct reals $a_1, \ldots, a_n$, and write down $2^n - 1$ numbers: $\sum_{i \in I} a_i$ for every nonempty subset $I \subseteq \{1, \ldots, n\}$. He then figures out $a_1, \ldots, a_n$. Prove that this is always possible if none of the sums you wrote down are equal to 0.