# Geometric Graph Theory

Exercise session 9, April 22, 2015

Solutions of selected problems are accepted until April 29, 2015.

If $P$ is a set of $n$ points in the plane in general position, a segment $s$ connecting two points of $P$ is a *halving segment* if each open halfplane determined by $s$ contains $\lfloor (n-2)/2 \rfloor$ or $\lceil (n-2)/2 \rceil$ points of $P$.

1. We have four random variables, $X_1, X_2, X_3, X_4$. We say that an event is *likely* if its chance of occurring is larger than 50%. If the events $X_1 > X_2$ and $X_3 > X_4$ are both likely, then is $X_1 + X_3 > X_2 + X_4$ also necessarily likely?

2. Prove that the number of incidences between $n$ points and $n$ circles in the plane is at most $O(n^{3/2})$.

3. Prove that if a graph with $n \geq 3$ vertices can be drawn in the plane so that every edge crosses at most one other edge, then it has at most $4n - 8$ edges.

4. Let $P$ be a set of an even number of points in the plane in general position, and let $v$ be a point in $P$. Show that the halving segments and their reflections through $v$ alternate around $v$. That is, between any pair of halving segments starting at $v$, there is a reflection of another halving segment starting at $v$.

✎ 5. Suppose that $n$ is even. Let $G$ be the geometric graph determined by the halving segments of $n$ points in the plane in general position. Let $c$ be the number of crossing in $G$. Prove that

$$c + \sum_{v \in V(G)} \binom{(d(v)+1)/2}{2} = \binom{n/2}{2}.$$

Advice: start with $n$ points in convex position and move them one by one to the vertices of $G$.

$ 6. A *dice* (or a *die*) is a cube with a positive integer written on each of its sides (different sides can have the same number). We say that a dice $A$ *is better* than a dice $B$ if the probability that after throwing $A$ and $B$ on the table, the number on top of $A$ is greater than the number on top of $B$, is more than 50%. Design three dice $A, B, C$ such that $A$ is better than $B$, $B$ is better than $C$, and $C$ is better than $A$.

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✎ — an optional homework. You may submit a written solution of this problem until the beginning of the next lecture and receive our feedback.

$ — an optional contest problem. You may submit a solution until the beginning of the next lecture. We encourage you to participate.

New exercises and notes can be found at `http://dcg.epfl.ch/page-117408-en.html`.