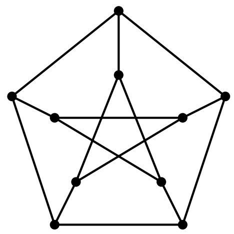
Geometric Graph Theory

Exercise session 1, 18. February 2015

1. Prove that for any connected graph G, we have $e(G) \geq v(G) - 1$ with equality if and only if G contains no cycles. Such graphs are called *trees*.

2. Prove that the complement of a planar graph with at least 11 vertices is not planar.

3. a) Show that the Petersen graph (in the Figure) is not planar.
   b) How many edges do we need to delete from it to make it planar?
   c) Does it matter which two?



4. Prove that any planar graph can be triangulated, that is, extended to a triangulation by adding new non-crossing edges to it.

5. Let G be a planar graph whose exterior face is a cycle.
   a) Prove that there are at least two vertices of this cycle that are connected to only two other vertices from the cycle.
   b) Can this be improved to three vertices if the cycle is large enough?

6. Let $\{p_1, \ldots, p_n\}$ be a set of $n \geq 3$ different points in the plane such that the smallest distance between them is 1.
   a) Show that the number of pairs which are at distance 1 from each other, is at most $3n - 6$.
   b) Is this bound tight? Why?
   c) Show that for any large enough n, there is a configuration with the smallest distance appearing at least $2.99n$ times.

7. * A finite number of straight lines divide the plane into polygonal regions (faces), some of which are unbounded.
   a) Show that these faces can be colored by two colors so that no two of them that share a side receive the same color.
   b) Can you tell which plane graphs have the property that their faces can be colored by two colors so that no two of them that share a side receive the same color?