Packing and covering

EPFL, 2014 Spring

List of theorems and statements

Unless stated otherwise, the knowledge of the *proofs* of the statements and theorems below is required; naturally, you should also be familiar with the corresponding definitions and properties.
Most of the material is covered (somewhat informally) by the lecture notes online. The main reference remains the volume "Combinatorial Geometry" by J. Pach and P. K. Agarwal, which can be found in the library. At the references, the first number (e.g. Theorem 1) refers to the numbering in the notes, followed by the reference to corresponding theorem in the book in parenthesis. Note that there are some proofs which are not contained in the book, only in the notes; in these instances, we only refer to the online notes.

Lecture 1: Lattices. Fundamental cell, independence of the determinant: Theorem 1. Lattice points close to each other: Theorem 2 (Theorem 1.3 - page 4). Maximal density of lattice packings of the unit disc: Corollary 1 (Corollary 1.4 - page 4). Minkowski's theorem: Theorem 3 (Theorem 1.7 - page 6). Lattice proof for Bézout's identity: Theorem 4 (proof in the online lecture notes). Pick's theorem (without proof).

Lecture 2: Polygons inscribed in convex discs. Dowker's theorems: Theorem 1 (Theorem 2.1 - page 11), Theorem 2 (Theorem 2.3 - page 13). Analogous statements for the perimeter - without proof: Theorem 3. Convex discs of maximal area inscribed in the unit disc: Proposition 1 (proof in the online lecture notes). Sas's theorem about largest inscribed polygons: Theorem 4 (Theorem 2.6 - page 14).

Lecture 3: Convex polytopes; Macbeath's theorem (only the statement): Theorem 1 (Theorem 2.10 - page 16). Elekes's theorem about upper bound for volume of polytopes inscribed in the unit ball: Theorem 2 (Theorem 2.11 - page 16).

Lecture 4: Definition of density of lattice packings. Approximation by polygons: Lemma 1 (Lemma 3.3, page 23). Jensen's inequality (without proof): Lemma 2 (in the online lecture notes). Extremality of hexagons: the theorem of L. Fejes Tóth: Theorem 2 (Theorem 3.2, page 22), Corollary 1 (Corollary 3.4, page 24). Thue's theorem: Theorem 1 (page 24, the remarks after Corollary 3.4).

Lecture 5: Density of packings of translates of a *centrally symmetric* convex disc: Lemma 1 (in the online lecture notes), Corollary 2 (in the online lecture notes), Corollary 3 (Corollary 3.6, page 25). Packings of similar copies of a convex disc: Theorem 2 (Theorem 3.7, page 25) - only the statement.

Lecture 6: Definition of crossing sets. Coverings with non-crossing convex discs: Lemma 1 (Lemma 3.9, page 27), Theorem 1 (Theorem 3.8, page 27), Corollary 1 (page 28). Estimates for the covering density of centrally symmetric convex discs: Corollary 2 (Corollary 3.12, page 29), Corollary 3 (Corollary 3.13, page 29).

Lecture 7: Definition of difference regions. Existence of inscribed affine regular hexagons: Lemma 1 (Lemma 4.3, page 38). The proof of Fáry's theorem (Theorem 4.1, p. 37). Definition of Dirichlet-Voronoi cells. Estimate on the number of sides: Corollary 1 (page 47; see Lemma 3.3, page 23.). Corollary 2 (see page 47). A new proof of Thue's theorem: Lemma 2 (Lemma 5.2, page 48).

Lecture 8: Further properties of Dirichlet-Voronoi cells. Definition of shadow cells. Method of cell decomposition: Lemma 1 (in the online lecture notes). Rogers' theorem about the optimality of lattice packings: Theorem 1 (Theorem 5.5, page 51). Difference sets: Lemma 2 (Lemma 5.6, page 51). Trigonal discs: Lemma 3 (Lemma 5.8, page 52).

Lecture 9: Sphere packings in high dimensions. Blichfeldt's method for estimating the density of sphere packings: Lemma 1 (Theorem 6.1., page 56), Blichfeldt's inequality: Lemma 2 (Lemma 6.2, page 57), Blichfeldt's theorem: Theorem 1 (Corollary 6.3, page 58).

Lecture 10: Definition of convex polytopes. Faces, the lattice of faces. Affine combinations, affine independence. Definition of the simplex, properties. The natural representation of the regular simplex. Estimate on the distance of faces from the center of Voronoi cells of a unit ball packing: Lemma 1 (Theorem 6.6, page 60).

Lecture 11: Proof of Rogers' simplex bound: Decomposition of Voronoi cells into simplices: Lemma 2 (Lemma 6.7, page 61); Rogers' theorem: Theorem 1 (Theorem 6.8, p. 62).

Lecture 12: Existence of dense lattice packings. Definition of admissible lattice, critical determinant. Mahler's theorem about the extremizers (without proof): Theorem 1 (in the online lecture notes). Davenport-Rogers lemma: Lemma 1 (Lemma 7.5, page 73). Hlawka's estimate: Theorem 2 (Theorem 7.6, page 74).

Lecture 13: The first part of the Minkowski-Hlawka theorem: Theorem 7.7, part i) (page 76). Packings of the difference body: Lemma 7.9 (page 78), Corollary 7.10 (page 78).

Lecture 14: The Rogers-Shephard inequality: Theorem 7.11 (page 78).