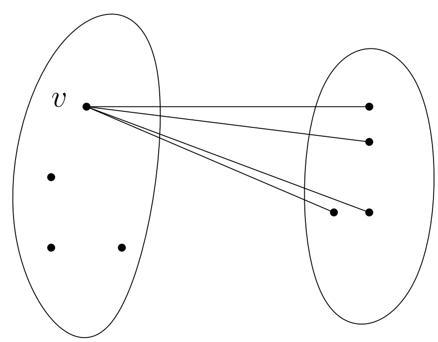
Graph Theory 2015 - theorem lecture 11

Theorem 1. *Let $G = (V, E)$ be a graph on n vertices, such that G does not contain K_3 as subgraph and $|E| = ex(n, K_3) - t$ for some non-negative integer t. Then one can remove at most t edges from G to make it bipartite.*

Proof. Let $v \in V$ be a vertex of maximum degree in G and let us denote its degree by Δ.

We denote by $N(v)$ the set of neighbors of v in G, and we define $B := N(v)$ and $A := V \backslash N(v)$. Since v is of maximum degree, we have that $|B| = \Delta$ and $|A| = n - \Delta$.

For any subgraphs H_1 and H_2 of G, we denote by $E(H_1, H_2)$ the set of edges of G with one extremity in H_1 and the other in H_2. We use the notation $E(H)$ for $E(H, H)$.



$A := V \backslash N(v)$ $\qquad$ $B := N(v)$

Let us first observe that there are no edges with both endpoints in B. Indeed, let us assume that there are two vertices $v_1, v_2 \in B$ such that (v_1, v_2) is an edge in G. Then the vertices v, v_1, v_2 form a K_3.

We prove now that $|E(A)| \leq t$. If this is indeed the case, the deletion of the edges in $E(A)$ will provide us a bipartite graph with bipartition $A \cup B$ on the same set of vertices as G (remember that there are no edges running between vertices of B, and after deletion there will be no edges running between vertices of A).

Since the cardinality of A is $n - \Delta$, and the degree of each vertex $u \in A$ is at most Δ, we have that

$$\Delta(n - \Delta) \geq \sum_{u \in A} d(u).$$

On the other hand, we have that

$$\sum_{u \in A} d(u) = 2|E(A)| + |E(A, B)| =$$

$$= |E(A)| + (|E(A) + |E(A, B)|) = |E(A)| + |E(G)|.$$

Since $|E(G)| = ex(n, 3) - t$, and the number of edges in $K_{\Delta, n-\Delta}$ (which is $\Delta(n - \Delta)$) cannot exceed $ex(n, K_3)$, we have that $|E(G)| \geq \Delta(n - \Delta) - t$.

This, together with the two inequalities above, imply that $|E(A)| \leq t$. We can now remove all the edges of $E(A)$ to obtain a bipartite graph, which completes the proof. $\square$