Introduction to Combinatorics Spring, 2011

Homework 9 – Probabilistic Methods

Janos Pach & Nabil Mustafa *21 April*

Questions

1. Let X be a number chosen uniformly at random from $\{1, \ldots, n\}$. Compute $Var[X]$. What if X is chosen from $\{-k, -k+1, \ldots, 0, \ldots, k-1, k\}$?

   Solution: $E[X] = \sum_{i=1}^{n} i/n = (n+1)/2$. Then $V[X] = \sum_{i=1}^{n} \frac{1}{n}(i-(n+1)/2)^2$, and simplify.

2. Suppose we roll a die 100 times. Let X be the sum of the numbers that appear over these 100 rolls. What is the best bound you can give for $Pr(|X - 350| \geq 50)$?

   Solution: Let X_i be the number appearing in the i-th roll. Then $E[X] = \sum E[X_i] = 100 \cdot 3.5 = 350$. To use Chebychev, need to compute variance: $V[X] = \sum_i V[X_i]$, where $V[X_i] = \sum_{j=1}^{6}(1/6)\cdot(j-3.5)^2$. Once we know the variance of X, use Chebychev, as $|X - 350|$ then is simply the distance from the mean $E[X] = 350$.

3. Given two independent random variables X and Y, prove that $Var[X - Y] = Var[X] + Var[Y]$.

   Solution: By algebra

$$\begin{aligned} V[X-Y] = E[(X-Y-E[X-Y])^2] &= E[((X-E[X]) - (Y-E[Y])^2] \\ &= E[(X-E[X])^2] + E[(Y-E[Y])^2] - 2E[X-E[X]]E[Y-E[Y]] \\ &= V[X] + V[Y] \end{aligned}$$

4. Construct a random variable to show that Markov's inequality is tight, i.e., given an integer k, construct a random variable X such that $Pr(X \geq k \cdot E[X]) = 1/k$.

   Solution: $X = k$ with probability $1/k$, and 0 otherwise. Calculation shows it satisfies required properties.

5. Similarly, can you give an example of a random variable to show that Chebychev's inequality is tight?

   Solution:

Bonus Problem. Given n red points $R = \{p_1, \ldots, p_n\}$ and n blue points $B = \{q_1, \ldots, q_n\}$ in the plane, prove that there is a one-to-one pairing of red points to blue points such that the n line segments in the plane (each defined by the two points in a pair) are disjoint.

10 points.